Full modal solution, asymptotic stability, reachability and observability

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Lec 7

Modal solution of CT system ZIR

$$\mathbf{q}(t) = \sum_{1}^{L} \alpha_i \mathbf{v}_i e^{\lambda_i t}$$

with the weights $\{\alpha_i\}_1^L$ determined by the initial condition:

$$\mathbf{q}(0) = \sum_{1}^{L} \alpha_i \mathbf{v}_i$$

Asymptotic stability of CT system

In order to have $\mathbf{q}(t) \to \mathbf{0}$ for all $\mathbf{q}(0)$, we require

$$\{Re(\lambda_i) < 0\}_1^L$$

i.e., all eigenvalues (natural frequencies) in open left half plane

The DT case: linearization at an equilibrium

$$\text{DT case:} \quad \mathbf{q}[n] = \bar{\mathbf{q}} + \widetilde{\mathbf{q}}[n] \ , \quad x[n] = \bar{x} + \widetilde{x}[n] \ ,$$

$$\mathbf{q}[n+1] = \mathbf{f}(\mathbf{q}[n], x[n])$$

$$\downarrow$$

$$\widetilde{\mathbf{q}}[n+1] \approx \left[\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\Big|_{\bar{\mathbf{q}},\bar{x}} \right] \widetilde{\mathbf{q}}[n] + \left[\frac{\partial \mathbf{f}}{\partial x}\Big|_{\bar{\mathbf{q}},\bar{x}} \right] \widetilde{x}[n]$$

for small perturbations $\widetilde{\mathbf{q}}[n]$ and $\widetilde{x}[n]$ from equilibrium

Modal solution of DT system ZIR

Could parallel CT development, but let's proceed differently:

$$\mathbf{A}\left[\,\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_L\,\right] = \left[\,\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_L\,\right] \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_L \end{bmatrix}$$

$$\text{or } \mathbf{AV} = \mathbf{V\Lambda}$$

$$\text{or } \mathbf{A} = \mathbf{V\Lambda V}^{-1}$$

$$\text{or } \mathbf{A}^n = (\mathbf{V\Lambda V}^{-1})\cdots(\mathbf{V\Lambda V}^{-1}) = \mathbf{V\Lambda}^n\mathbf{V}^{-1}$$

$$\mathbf{q}[n] = \mathbf{A}^n \mathbf{q}[0] = \mathbf{V}\boldsymbol{\Lambda}^n \underbrace{\mathbf{V}^{-1}\mathbf{q}[0]}_{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_L \end{bmatrix}}$$

$$\text{so } \mathbf{q}[n] = \left[\, \mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_L \,\right] \begin{bmatrix} \lambda_1^n & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^n & 0 & \cdots & 0 \\ \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_L^n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_L \end{bmatrix}$$

$$= \sum_{1}^{L} \alpha_i \mathbf{v}_i \ \ {}_{i}^{n}$$

Asymptotic stability of DT system

In order to have $\mathbf{q}[n] \to \mathbf{0}$ for all $\mathbf{q}[0]$, we require

$$\{|\lambda_i| < 1\}_1^L$$

i.e., all eigenvalues (natural frequencies)
inside unit circle

A^n for increasing n

$$\mathbf{A}_1 = \begin{bmatrix} 0.6 & 0.6 \\ 0.6 & 0.6 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 101 & 100 \\ -101 & -100 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 100.5 & 100 \\ -100.5 & -100 \end{bmatrix}, \qquad \mathbf{A}_4 = \begin{bmatrix} 0.6 & 100 \\ 0 & 0.5 \end{bmatrix}.$$

A^n for increasing n

$$\mathbf{A}_1^n = \begin{bmatrix} 0.6 & 0.6 \\ 0.6 & 0.6 \end{bmatrix}^n = (1.2)^n \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$\mathbf{A}_2^n = \begin{bmatrix} 101 & 100 \\ -101 & -100 \end{bmatrix}^n = \mathbf{A}_2$$

A^n for increasing n

$$\mathbf{A}_3^n = \begin{bmatrix} 100.5 & 100 \\ -100.5 & -100 \end{bmatrix}^n = (0.5)^n \begin{bmatrix} 201 & 200 \\ -201 & -200 \end{bmatrix}$$

$$\mathbf{A}_4^n = \begin{bmatrix} 0.6 & 100 \\ 0 & 0.5 \end{bmatrix}^n = \begin{bmatrix} 0.6^n & 1000(0.6^n - 0.5^n) \\ 0 & 0.5^n \end{bmatrix}$$

Modal solution of driven DT system

$$\mathbf{q}[n+1] = \mathbf{V}\mathbf{\Lambda}\underbrace{\mathbf{V}^{-1}\mathbf{q}[n]}_{\mathbf{r}[n]} + \mathbf{b}x[n] \ , \qquad y[n] = \mathbf{c}^T\mathbf{q}[n] + \mathtt{d}x[n]$$

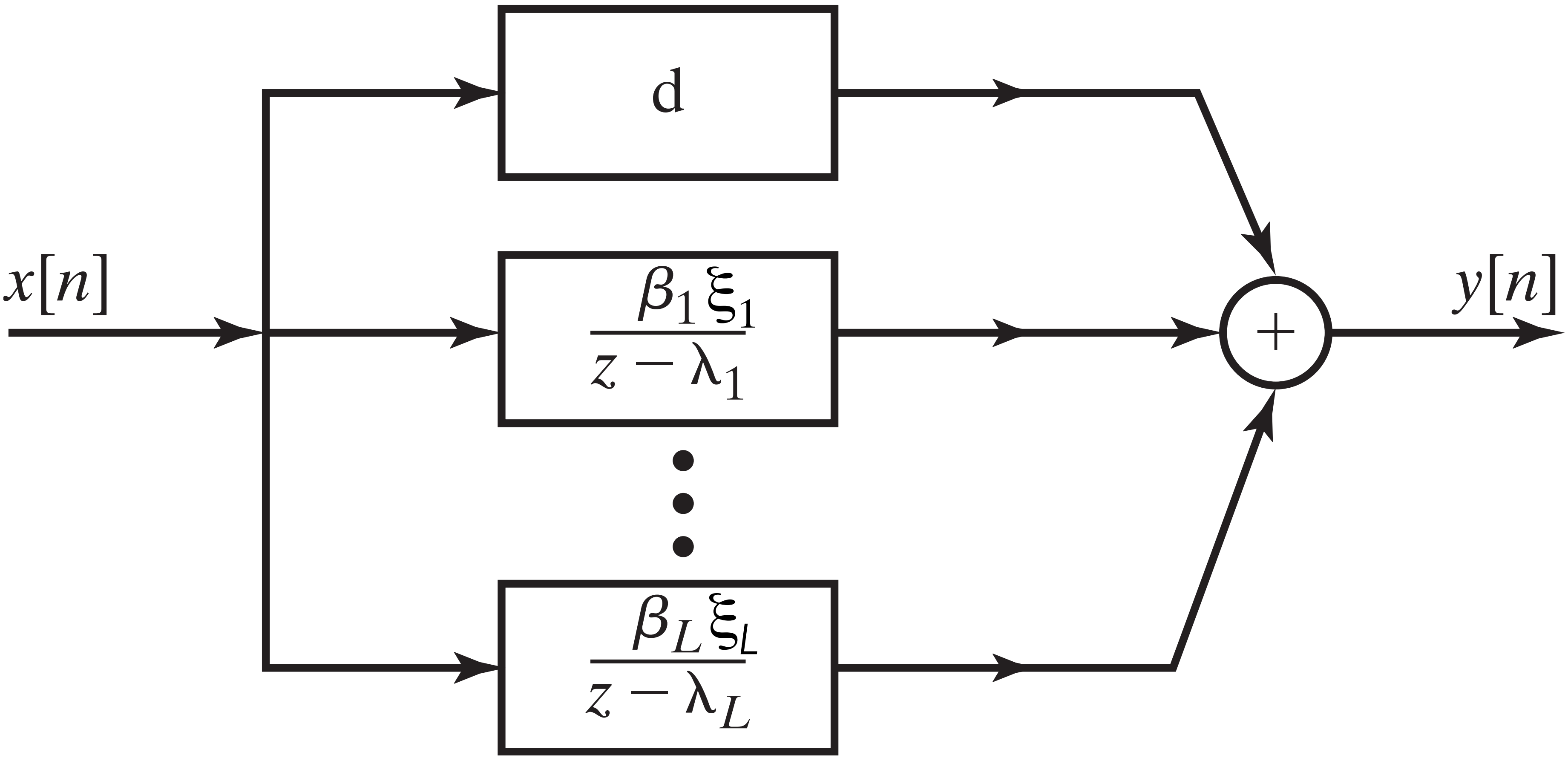
$$\downarrow$$

$$\mathbf{r}[n+1] = \mathbf{\Lambda}\mathbf{r}[n] + \underbrace{\mathbf{V}^{-1}\mathbf{b}}_{\beta} x[n] \ , \qquad y[n] = \underbrace{\mathbf{c}^T\mathbf{V}}_{\xi^T}\mathbf{r}[n] + \mathtt{d}x[n]$$

Because $\mathbf{\Lambda}$ is diagonal, we get the decoupled scalar equations

$$r_i[n+1] = \lambda_i r_i[n] + \beta_i x[n] \ , \qquad y[n] = \left(\sum_{1}^{L} \xi_i r_i[n]\right) + \mathtt{d}[n]$$

Underlying structure of LTI DT state-space system with L distinct modes



Reachability and Observability

$$r_i[n+1] = \lambda_i r_i[n] + \beta_i x[n] \ , \qquad y[n] = \Big(\sum_1^L \xi_i r_i[n]\Big) + \mathtt{d}[n]$$

for $i = 1, 2, \ldots, L$

$\downarrow$

$\beta_j = 0$, the jth mode cannot be excited from the input

i.e., the jth mode is **unreachable**

$\xi_k = 0$, the kth mode cannot be seen in the output

i.e., the kth mode is **unobservable**