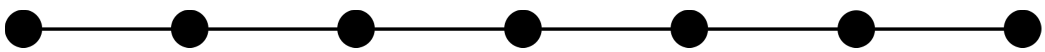
Massachusetts Institute of Technology
Departments of Electrical Engineering & Computer Science and Economics

6.207/14.15 Networks
Spring 2018

Problem Set 3

Problem 3.1
[P 11.3 Newman] Consider a "line graph" consisting of n vertices in a line like this:

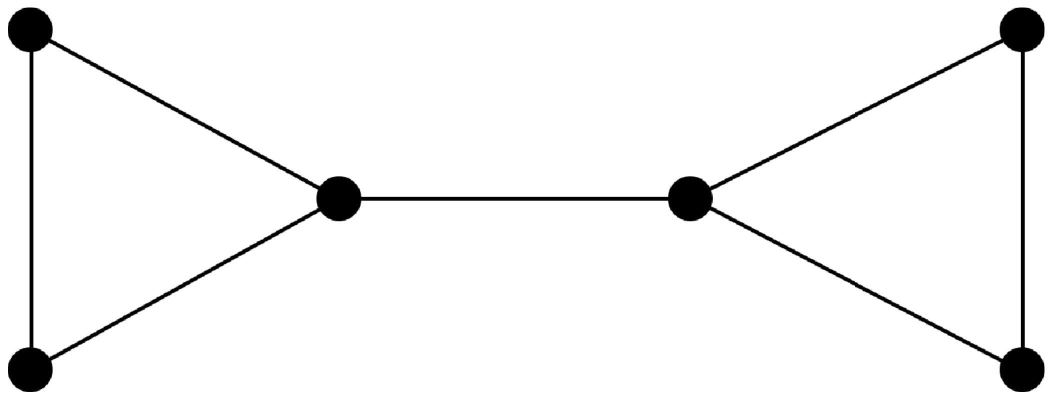


(a) Show that if we divide the network into two parts by cutting any single edge, such that one part has r vertices and the other has $n - r$, the modularity, Eq.(7.76), takes the value:

$$Q = \frac{3 - 4n + 4rn - 4r^2}{2(n-1)^2} \tag{1}$$

(b) Hence show that when n is even the optimal such division, in terms of modularity, is the division that splits the network exactly down the middle.

Problem 3.2
[P 11.4 Newman] Using your favorite numerical software for finding eigenvectors of matrices, construct the Laplacian and modularity matrix for this small network:



(a) Find the eigenvector of the Laplacian corresponding to the second smallest eigenvalue and hence perform a spectral bisection of the network into two equally sized parts.

(b) Find the eigenvector of the modularity matrix corresponding to the largest eigenvalue and hence divide the network into two communities.

You should find that the division of the network generated by the two methods is in this case, the same.

Problem 3.3
Consider an Erdös-Renyi random graph $G(n, p)$

(a) Let A_1 denote the event that node 1 has at least $l \in \mathbb{Z}^+$ neighbhors. Do we observe a phase transition for this event? If so, find the threshold function and explain your reasoning.

(b) Let B denote the event that a cycle with k edges (for a fixed k) emerges in the graph. Do we observe a phase transition of this event? If so, find the threshold function and explain your reasoning.

Problem 3.4
[P 12.6 Newman] We can make a simple random graph model of a network with clustering or transitivity as follows. We take n vertices and go through

each distinct trio of three vertices, of which there are $\binom{n}{3}$, and with independent probability p we connect the members of the trio together using three edges to form a triangle, where $p = \frac{c}{\binom{n-1}{2}}$ with c constant.

(a) Show that the mean degree of a vertex in this network is $2c$.

(b) Show that the degree distribution is

$$p_k = \begin{cases} e^{-c}c^{k/2}/(k/2)! & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

(c) Show that the clustering coefficient is $C = \frac{1}{2c+1}$.

(d) Show that when there is a giant component in the network, its expected size S, as a fraction of the network size, satisfies $S = 1 - e^{-cS(2-S)}$.

(e) What is the value of the clustering coefficient when the giant component fills half of the network?

Problem 3.5

Consider the variation on the small-world model proposed by Newman and Watts: consider a ring lattice with n nodes in which each node is connected to its neighbors k hops or less away. For each edge, with one probability p, add a new edge to the ring lattice between two nodes chosen uniformly at random.

(a) Find the degree distribution of this model.

(b) Show that when $p = 0$, the overall clustering coefficient of this graph is given by

$$Cl(g) = \frac{3k-3}{4k-2}.$$

(c) *(Optional for Bonus)*: Show that when $p > 0$, the overall clustering coefficient is given by

$$Cl(g) = \frac{3k-3}{4k-2}(1-p)^3.$$

Problem 3.6

Consider a society of n individuals. A randomly chosen node is infected with a contagious infection. Assume that the network of interactions in the society is represented by a configuration model with degree distribution p_k. Assume that any individual is immune independently with probability π. We would like to investigate whether the infection can spread to a nontrivial fraction of the society.

(a) Find a threshold for the immunity probability (in terms of the moments of the degree distribution) below which the infection spreads to a large portion of the population.

(b) What is this threshold for a k-regular random graph, i.e., a configuration model network in which all nodes have the same degree.

(c) What is this threshold for a power law graph with exponents less than 3, i.e., $p_k \sim k^{-\alpha}$ with $\alpha < 3$? The Internet graph (representing connections between routers) has a power law distribution with exponent $\sim 2.1 - 2.7$. What does this result imply for the Internet graph?

(d) Find the size of the infected population (you can assume that the infection spreads to a large portion of the population).