# Lectures 6–7: Monotone Comparative Statics, with Applications to Producer Theory

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# The Plan for this Week

Traditional topic to cover after neoclassical consumer theory is **neoclassical producer theory**:
theory of profit-maximizing production choices by a firm.

These theories are extremely similar.

**Plan:**

- Super-fast treatment of neoclassical producer theory.
- Spend time covering critical modern tool for economic analysis: **monotone comparative statics**.
- Illustrate MCS with applications to some modern results in producer theory.

# Neoclassical Producer Theory in One Sentence

"Producers are just like consumers, but they maximize profit instead of utility."

We expand on this just slightly, and show how main results of producer theory follow from results from consumer theory.

# The Profit Maximization Problem (PMP)

Choose **production plan** $y \in \mathbb{R}^n$ from **production possibilities set** $Y \subseteq \mathbb{R}^n$ to maximize **profit** $p \cdot y$:

$$\max_{y \in Y} p \cdot y$$

- Some prices can be negative.
  Lets us model inputs and outpus symmetrically.
  - Inputs have negative prices (firm pays to use them).
  - Outputs have positive prices (firm makes money by producing).
- Neoclassical firm is **price taker**.
  - No market power.
  - Study of firms with market power is a topic in **industrial organization**.
- Firm's objective is **profit maximization**.
  - In reality, firm is organization composed of individuals with different goals.
  - Study of internal behavior and organization of firms is topic in **organizational economics**.

# The PMP and the EMP

For our purposes, producer theory leaves everything interesting about firm behavior to other areas of economics, and reduces firm's problem to something isomorphic to consumer's expenditure minimization problem.

PMP is

$$\max_{y \in Y} p \cdot y.$$

Letting $S = \{x \in \mathbb{R}^n : u(x) \geq u\}$, EMP is

$$\min_{x \in S} p \cdot x.$$

Up to flipping a sign, PMP the same as EMP.

EMP: consumer chooses bundle of goods $x$ to minimize expenditure, subject to $x$ lying in set $S$.

PMP: firm chooses bundle of goods $y$ to minimize *net* expenditure (maximize net profit), subject to $y$ lying in set $Y$.

# The PMP and the EMP

Solution to EMP: Hicksian demand $h(p)$.
Value function for EMP: expenditure function $e(p)$.
(omitting $u$ because we hold it fixed)

Solution to PMP: **optimal production plan** $y(p)$.
Value function for EMP: **profit function** $\pi(p)$.

Our treatment of producer theory consists of recalling facts about Hicksian demand and expenditure function, and translating into language of optimal production plan and profit function.

# Properties of Hicksian Demand/Optimal Production Plans

**Theorem**

*Hicksian demand satisfies:*

1. *Homogeneity of degree* 0*: for all* $\lambda > 0$, $h(\lambda p) = h(p)$.
2. *Convexity: if* $S$ *is convex (i.e., if preferences are convex), then* $h(p)$ *is a convex set.*
3. *Law of demand: for every* $p, p' \in \mathbb{R}^n$, $x \in h(p)$, *and* $x' \in h(p')$, *we have* $(p' - p)(x' - x) \leq 0$.

**Theorem**

*Optimal production plans satisfy:*

1. *Homogeneity of degree* 0*: for all* $\lambda > 0$, $y(\lambda p) = y(p)$.
2. *Convexity: if* $Y$ *is convex, then* $y(p)$ *is a convex set.*
3. *Law of supply: for every* $p, p' \in \mathbb{R}^n$, $y \in y(p)$, *and* $y' \in y(p')$, *we have* $(p' - p)(y' - y) \geq 0$.

# Properties of Expenditure Function/Profit Function

## Theorem

*The expenditure function satisfies:*

1. *Homogeneity of degree* 1*: for all* $\lambda > 0$, $e(\lambda p) = \lambda e(p)$.
2. *Monotonicity:* $e$ *is non-decreasing in* $p$.
3. *Concavity:* $e$ *is concave in* $p$.
4. *Shephard's lemma: under mild conditions (see Lectures 2–3),* $e$ *is differentiable, and* $\frac{\partial}{\partial p_i} e(p) = h_i(p)$.

## Theorem

*The profit function satisfies:*

1. *Homogeneity of degree* 1*: for all* $\lambda > 0$, $\pi(\lambda p) = \lambda \pi(p)$.
2. *Monotonicity:* $\pi$ *is non-decreasing in* $p$.
3. *Convexity:* $\pi$ *is convex in* $p$.
4. *Hotelling's lemma: under mild conditions,* $\pi$ *is differentiable, and* $\frac{\partial}{\partial p_i} \pi(p) = y_i(p)$.

# Monotone Comparative Statics: Motivation

**Comparative statics** are statements about how solution to a problem changes with parameters.

Core of most applied economic analysis.

Last twenty years or so:
revolution in how comparative statics are done in economics.

Traditional approach:
differentiate FOC using implicit function theorem.

New approach:
monotone comparative statics.

# Example: Traditional Approach

Consider problem:

$$\max_{x \in X} b\left(x, \theta\right) - c\left(x\right)$$

- $x$ is choice variable
- $\theta$ is parameter
- $b\left(x, \theta\right)$ is benefit from choosing $x$ given parameter $\theta$
- $c$ is cost of choosing $x$

# Example: Traditional Approach

$$\max_{x \in X} b(x, \theta) - c(x)$$

If $X \subseteq \mathbb{R}$ and $b$ and $c$ are differentiable, FOC is

$$b_x(x^*(\theta), \theta) = c'(x^*(\theta)).$$

If $b$ and $c$ are twice continuously differentiable and $b_{xx}(x^*(\theta), \theta) \neq c''(x^*(\theta))$, implicit function theorem implies that solution $x^*(\theta)$ is continuously differentiable, with derivative

$$\frac{d}{d\theta} x^*(\theta) = \frac{b_{x\theta}(x^*(\theta), \theta)}{c''(x^*(\theta)) - b_{xx}(x^*(\theta), \theta)}.$$

If $c$ is convex, $b$ is concave in $x$, and $b_{x\theta} > 0$, can conclude that $x^*(\theta)$ is (locally) increasing in $\theta$.
Intuition: FOC sets marginal benefit equal to marginal cost.
If $b_{x\theta} > 0$ and $\theta$ increases, then if $b$ is concave in $x$ and $c$ is convex, $x$ must increase to keep the FOC satisfied.

# What's Wrong with This Picture?

**Unnecessary assumptions:** as we'll see, solution(s) are increasing in $\theta$ even if $b$ is **not concave**, $c$ is **not convex**, $b$ and $c$ are **not differentiable**, and choice variable is **not continuous or real-valued**.

**Wrong intuition**: Intuition coming from the FOC involves concavity of $b$ and convexity of $c$.
This can't be the right intuition.

We'll see that what's really needed is an ordinal condition on $b$—the **single-crossing property**—which is a more meaningful version of the assumption $b_{x\theta} > 0$.

# Why Learn Monotone Comparative Statics?

Three reasons:

1. **Generality:** Cut unnecessary convexity and differentiability assumptions.
2. **Analytical power:** Often, can't assume convexity and differentiability.
   (Traditional approach doesn't work.)
3. **Understanding:** By focusing on essential assumptions, help to understand workings of economic models.
   (Don't get confused about what drives what.)

# Why Learn Monotone Comparative Statics?

Fourth reason: need to understand them to read other people's papers.

- Costinot, A. "An Elementary Theory of Comparative Advantage." *Econometrica*, 2009. [International]
- Acemoglu, D. "When Does Labor Scarcity Encourage Innovation?" *Journal of Political Economy*, 2010. [Growth/Innovation]
- Kircher, P. and J. Eeckhout. "Sorting and Decentralized Price Competition." *Econometrica*, 2010. [Labor]
- Segal, I. and M. Whinston. "Property Rights." Chapter for Handbook of Organizational Economics, 2011. [Organizational Econ]
- Acemoglu, D. and A. Wolitzky. "The Economics of Labor Coercion." *Econometrica*, 2011. [Political Economy]

# MCS with 1 Choice Variable and 1 Parameter

Start with simple case: $X \subseteq \mathbb{R}$, $\Theta \subseteq \mathbb{R}$.

Interested in set of solutions $X^* (\theta)$ to optimization problem

$$\max_{x \in X} f(x, \theta)$$

Under what conditions on $f$ is $X^* (\theta)$ increasing in $\theta$?

# The Strong Set Order

What does it mean for **set** of solutions to be increasing?

Relevant order on sets: **strong set order**.

## Definition

A set $A \subseteq \mathbb{R}$ is greater than a set $B \subseteq \mathbb{R}$ in the **strong set order (SSO)** if, for any $a \in A$ and $b \in B$,

$$\begin{aligned} \max\{a, b\} &\in A, \text{ and} \\ \min\{a, b\} &\in B. \end{aligned}$$

$X^*(\theta)$ greater than $X^*(\theta')$ if, whenever $x$ is solution at $\theta$ and $x'$ is solution at $\theta'$, either

1. $x \geq x'$, or
2. both $x$ and $x'$ are solutions for both parameters.

# Increasing Differences

Simple condition on $f$ that guarantees that $X^*(\theta)$ is increasing (in SSO): **increasing differences**.

## Definition

A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has **increasing differences** in $(x, \theta)$ if, whenever $x^H \geq x^L$ and $\theta^H \geq \theta^L$, we have

$$f\left(x^H, \theta^H\right) - f\left(x^L, \theta^H\right) \geq f\left(x^H, \theta^L\right) - f\left(x^L, \theta^L\right).$$

Return to choosing a higher value of $x$ is non-decreasing in $\theta$.

Form of complementarity between $x$ and $\theta$.

# Increasing Differences: Differential Version

**Theorem**

*If $f$ is twice continuously differentiable, then $f$ has increasing differences in $(x, \theta)$ iff*

$$\frac{\partial^2 f(x, \theta)}{\partial x \partial \theta} \geq 0 \text{ for all } x \in X, \theta \in \Theta.$$

Increasing differences generalizes condition on cross-partial derivatives used to sign comparative statics in traditional approach.

# Topkis' Monotonicity Theorem

Simplest MCS theorem:

**Theorem (Topkis)**

*If $f$ has increasing differences in $(x, \theta)$, then $X^*(\theta)$ is increasing in the strong set order.*

# Back to Example

$$\max_{x \in X} b\left(x, \theta\right) - c\left(x\right)$$

If $b$ has increasing differences in $\left(x, \theta\right)$, then $X^*\left(\theta\right)$ is increasing in the strong set order.

No assumptions about convexity or differentiability of anything.

# Necessity

Want to find minimal assumptions for given comparative statics result to hold.

Is increasing differences minimal assumption?

**No:** increasing differences is cardinal property, but property that $X^*(\theta)$ is increasing is ordinal.

What's ordinal version of increasing differences?

# Single-Crossing

## Definition

A function $f : X \times \Theta \rightarrow \mathbb{R}$ is **single-crossing** in $(x, \theta)$ if, whenever $x^H \geq x^L$ and $\theta^H \geq \theta^L$, we have

$$f\left(x^H, \theta^L\right) \geq f\left(x^L, \theta^L\right) \implies f\left(x^H, \theta^H\right) \geq f\left(x^L, \theta^H\right)$$

and

$$f\left(x^H, \theta^L\right) > f\left(x^L, \theta^L\right) \implies f\left(x^H, \theta^H\right) > f\left(x^L, \theta^H\right).$$

Whenever choosing a higher $x$ is better at a low value of $\theta$, it's also better at a high value of $\theta$.

Increasing differences implies single-crossing, but not vice versa.

# Milgrom-Shannon Monotonicity Theorem

## Theorem (Milgrom and Shannon)

*If $f$ is single-crossing in $(x, \theta)$, then $X^* (\theta)$ is increasing in the strong set order.*
*Conversely, if $X^* (\theta)$ is increasing in the strong set order for* **every** *choice set $X \subseteq \mathbb{R}$, then $f$ is single-crossing in $(x, \theta)$.*

# Strictly Increasing Selections

A stronger set order: for $\theta < \theta'$, every $x \in X^*(\theta)$ is strictly less than every $x' \in X^*(\theta')$.
(Every selection is strictly increasing.)

When is every selection strictly increasing?

**Strictly increasing differences**: whenever $x^H > x^L$ and $\theta^H > \theta^L$, we have

$$f\left(x^H, \theta^H\right) - f\left(x^L, \theta^H\right) > f\left(x^H, \theta^L\right) - f\left(x^L, \theta^L\right).$$

## Theorem (Edlin and Shannon)

*Suppose $f$ is continuously differentiable in $x$ and has strictly increasing differences in $(x, \theta)$.*
*Then, for all $\theta < \theta'$, $x^* \in X^*(\theta) \cap \operatorname{int} X$, and $x^{*\prime} \in X^*(\theta')$, we have $x^* < x^{*\prime}$.*

# MCS with n Choice Variables and m Parameters

Previous theorems generalize to $X \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^m$.

Two main issues in generalization:

1. What's "max" or "min" of two vectors?
2. Need complementarity within components of $x$, not just between $x$ and $\theta$.

Once clear these up, analysis same as in 1-dimensional case.

# Meet and Join

Relevant notion of min and max are **component-wise** min and max, also called **meet** and **join**:

$$\begin{aligned} x \wedge y &= \left(\min\left\{x_1, y_1\right\}, \ldots, \min\left\{x_n, y_n\right\}\right) \\ x \vee y &= \left(\max\left\{x_1, y_1\right\}, \ldots, \max\left\{x_n, y_n\right\}\right) \end{aligned}$$

## Definition

A set $A \subseteq \mathbb{R}^n$ is greater than a set $B \subseteq \mathbb{R}^n$ in the **strong set order** if, for any $a \in A$ and $b \in B$,

$$\begin{aligned} a \vee b &\in A, \text{ and} \\ a \wedge b &\in B. \end{aligned}$$

A **lattice** is a set $X \subseteq \mathbb{R}^n$ such that $x \wedge y \in X$ and $x \vee y \in X$ for all $x, y \in X$.

Ex. A product set $X = X_1 \times \ldots X_n$ is a lattice.

# Increasing Differences

Definition of increasing differences in $(x, \theta)$ same as before:
$x^H \geq x^L, \theta^H \geq \theta^L \implies$

$$f\left(x^H, \theta^H\right) - f\left(x^L, \theta^H\right) \geq f\left(x^H, \theta^L\right) - f\left(x^L, \theta^L\right)$$

(Note: $x$ and $\theta$ are vectors. What does $x^H \geq x^L$ mean?)

Increasing differences in $(x, \theta)$ no longer enough to guarantee $X^*(\theta)$ increasing.

Issue: what if increase in $\theta_1$ pushes $x_1$ and $x_2$ up, but increase in $x_1$ pushes $x_2$ *down*?

Need complementarity within components of $x$, not just between $x$ and $\theta$.

This is called **supermodularity** of $f$ in $x$.

# Supermodularity

## Definition

A function $f : X \times \Theta \to \mathbb{R}$ is **supermodular** in $x$ if, for all $x, y \in X$ and $\theta \in \Theta$, we have

$$f\left(x \vee y, \theta\right) - f\left(x, \theta\right) \geq f\left(y, \theta\right) - f\left(x \wedge y, \theta\right).$$

# Differential Versions

**Theorem**

*If $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is twice continuously differentiable, then $f$ has increasing differences in $(x, \theta)$ iff*

$$\frac{\partial^2 f(x,\theta)}{\partial x_i \partial \theta_j} \geq 0 \text{ for all } x \in X, \theta \in \Theta, i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\},$$

*and $f$ is supermodular in $x$ iff*

$$\frac{\partial^2 f(x,\theta)}{\partial x_i \partial x_j} \geq 0 \text{ for all } x \in X, \theta \in \Theta, i \neq j \in \{1, \ldots, n\}.$$

# Topkis' Theorem

### Theorem

*If* $X \subseteq \mathbb{R}^n$ *is a lattice,* $\Theta \subseteq \mathbb{R}^m$*, and* $f : X \times \Theta \to \mathbb{R}$ *has increasing differences in* $(x, \theta)$ *and is supermodular in* $x$*, then* $X^* (\theta)$ *is increasing in the strong set order.*

There are also multidimensional versions of the Milgrom-Shannon and Edlin-Shannon theorems.

# Application 1: Comparative Statics of Input Utilization

Suppose firm has production function $f : \mathbb{R}^n \to \mathbb{R}$, output price $p$, input price vector $q$:

$$\max_{y \in \mathbb{R}^n_+} pf(y) - q \cdot y$$

Assume $f$ non-decreasing and supermodular.

$f$ non-decreasing $\implies$ objective has increasing differences in $(y, (p, -q))$.

### Theorem

*Suppose a competitive firm's production function is increasing and supermodular in its inputs. If the price of the firm's output increases and/or the price of* **any** *of its inputs decreases, then the firm increases the usage of* **all** *of its inputs.*
*(Formally, $Y^*(p, q)$ increases in the strong set order.)*

# Application 1.5: The Law of Supply

$$\max_{y \in \mathbb{R}^n_+} pf(y) - q \cdot y$$

Can use Topkis' theorem to give alternative proof of law of supply, without any assumptions on $f$.

Let

$$\begin{aligned} x &= f(y) \\ c(x) &= \min_{y \in \mathbb{R}^n_+ : f(y) \geq x} q \cdot y \end{aligned}$$

Rewrite problem as

$$\max_{x \in \mathbb{R}} px - c(x)$$

Problem has increasing differences in $(x, p)$, so $x^*(p)$ increasing in strong set order.
(And every selection from $x^*(p)$ is increasing: see pset.)

# Application 2: The LeChatelier Principle

"Firms react more to input price changes in the long-run than the short-run."

Suppose inputs are labor and capital, and capital is fixed in short run.

Seems reasonable that if price of labor changes, firm only adjusts labor slightly in short run, stuck with its old capital usage.

In long run, will adjust labor more, once can choose "right" capital usage.

We give example that shows LeChatelier Principle doesn't always apply, and then use Tokpis to formulate rigorous version of the principle.

# Example

Firm can produce \$10 of output by using **either**

1. 2 units of $L$.
2. 1 unit each of $L$ and $K$.
   Can also shut down and produce nothing.

Initial prices: \$2 per unit of $L$, \$3 per unit of $K$.

Firm produces using 2 units of $L$.

# Example (cnt.)

Suppose price of $L$ rises to \$6, $K$ fixed in short run.

In short run, firm shuts down.

In long run, firm produces using 1 unit each of $L$ and $K$.

In short run, demand for $L$ drops from 2 to 0.
In long run, goes back up to 1.
LeChatelier principle fails.

# Example (cnt.)

What went wrong?

1 unit of $L$ is complementary with 1 unit of $K$, but 2 units of $L$ are substitutable with 1 unit of $K$.

$L$ usage drops from 2 to 0 makes 1 unit of $K$ more valuable ("substitution"), but when $K$ usage rises from 0 to 1 this makes 1 unit of $L$ more valuable ("complementarity").

Suggests LeChatelier principle failed because inputs switched from being complements to substitues at different usage levels.

# LeChatelier Revisited

Let

$$\begin{aligned} x(y,\theta) &= \arg\max_{x\in X} f(x,y,\theta) \\ y(\theta) &= \arg\max_{y\in Y} f(x(y,\theta),y,\theta) \end{aligned}$$

$x(y,\theta)$ is optimal "short-run" $x$ (i.e., holding $y$ fixed).
$y(\theta)$ and $x(y(\theta),\theta)$ are optimal "long-run" choices.

**Theorem**

*Suppose $f : X \times Y \times \Theta \to \mathbb{R}$ is supermodular, $\theta \geq \theta'$, and maximizers below are unique. Then*

$$x(y(\theta),\theta) \geq x\left(y\left(\theta'\right),\theta\right) \geq x\left(y\left(\theta'\right),\theta'\right).$$

# LeChatelier Revisited

### Corollary (LeChatelier Principle)

*Suppose a firm's problem is*

$$\max_{K,L\in\mathbb{R}_+} pf(K,L) - wL - rK$$

*with either $f_{KL} \geq 0$ for all $(K,L)$ or $f_{KL} \leq 0$ for all $(K,L)$, and suppose $K$ is fixed in the short-run, while $L$ is flexible.*
*Then, if the wage $w$ increases, the firm's labor usage decreases, and the decrease is larger in the long-run than in the short-run.*