14.05 Intermediate Macro
Pset 4

Due April 19th

Problem 1: Learning by Doing with Spillovers

Consider the model of learning by doing with spillovers (Arrow & Romer) presented in class and assume that the production function is Cobb-Douglas, that is,

$$Y_t^m = (K_t^m)^\alpha (h_t L_t^m)^{1-\alpha}$$

However, assume there are diminishing returns to technological progress, $h_t = \eta k_t^\gamma$, for some constants $\eta > 0$, $0 < \gamma < 1$, where $k_t = \frac{K_t^m}{L_t^m}$.

i. We want to write the equilibrium dynamics are functions of c and k alone:

   (a) Express the return R that firms are willing to pay in equilibrium as a function of kt alone.

   (b) Express the resource constraint in terms of c and k.

ii. Imagine the continuous time version of the dynamics in part (a) and draw the phase diagram.

iii. Repeat parts (a) and (b) for the social planner's problem (Hint: this is similar to the Ramsey model).

iv. How does the phase diagram of part (c) compare to that of part (b)? which line changes, the $\dot{c} = 0$ locus or the $\dot{k} = 0$ locus? What happens to the steady state levels of c and k?

v. If the equilibrium allocations differ from the planner's allocations, describe a policy that would restore efficiency.

Problem 2: Tax smoothing

Consider a two-period economy. Households preferences are given by

$$U = u(c_1, c_2, n_1, n_2) = c_1 - n_1^2 + \beta\left(c_2 - n_2^2\right),$$

where $c_t \geq 0$ is consumption in period $t \in \{1, 2\}$ and $n_t \geq 0$ is labor supply. Labor is used to produce output with the technology $y_t = An_t$ (there is no capital). The wage is thus given by $w_t = A$, for $t \in \{1, 2\}$. The government taxes labor income at rates τ_t in period t, so households' intertemporal budget constraint is given by

$$c_1 + \frac{1}{1+r}c_2 = (1-\tau_1)An_1 + \frac{1}{1+r}(1-\tau_2)An_2$$

The government has constant expenditues, $g_t = g$ for $t \in \{1, 2\}$. Its intertemporal budget constraint is thus given by

$$IBC \equiv (\tau_1 An_1 - g_1) + \frac{1}{1+r}(\tau_2 An_2 - g_2) = 0$$

Finally, the resource constraints in the economy are $y_1 = An_1 = c_1 + g$ and $y_2 = An_2 = c_2 + g$.

1) Consider the household's optimal consumption and labor-supply problem. Argue that the solution is interior only if the interest rate r is such that $\frac{1}{1+r} = \beta$. Assume that this is the case for the rest of the exercise.

2) Solve for the household's optimal n_1 and n_2 as functions of τ_1 and τ_2.

3) Use the two resource constraints to replace $c_t = An_t - g$ into U. Next, use the previous result to replace n_t with a function of τ_t. You should now have expressed the household's utility U as a function of the two tax rates:

$$U = U(\tau_1, \tau_2)$$

4) Do the same for the government's intertemporal buget: replace n_t with the function of τ_t that you found in part 2 so as to express IBC in terms of τ_1 and τ_2 :

$$IBC = IBC(\tau_1, \tau_2)$$

5) It follows that the optimal policy is given by the combination of τ_1 and τ_2 that solves the following problem:

$$\max U(\tau_1, \tau_2)$$
$$s.t. \quad IBC(\tau_1, \tau_2) = 0$$

Prove that the optimal policy satisfies $\tau_1 = \tau_2$ (tax smoothing).

6) Suppose that we increase g_1 but reduce g_2 so that $g_1 + \beta g_2$ stays constant. What happens to the optimal taxes? Explain.