Chapter 4

'Statics' of a rotating system

Supplemental reading:

Holton (1979), chapter 1 and section 3.2

Houghton (1977), sections 7.3–7.6

4.1 Geostrophy and hydrostaticity

In this chapter some terms are introduced which are needed for the subsequent review of observations. Specifically, we will review hydrostaticity and introduce geostrophy – both involve static balances[1]. Our approach will be 'quick and dirty'. Matters will be approached more carefully in Chapter 6. In static balances, forces are balanced by pressure gradients: $-\nabla p$. Consider a rotating fluid in a gravitational field (oriented parallel to the rotation vector; *viz.* Figure 4.1).

Let the flow consist simply in rotation about the z-axis: $\Omega + \omega(r)$, where $r = \sqrt{x^2 + y^2}$. Moreover, let the flow be steady. The force balance in the z-direction will simply be

$$\frac{\partial p}{\partial z} = -\rho g. \tag{4.1}$$

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[1]It is expected that most readers will be more or less familiar with the contents of this chapter. It is included for the few who may have need of it. Nevertheless, the material is essential. The reader is urged to make sure that all the exercises that follow this chapter can be comfortably handled.
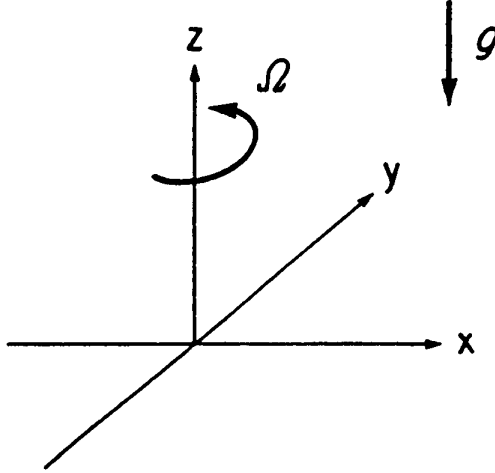
Figure 4.1: Orientation of rotation and gravity in cartesian frame.

This balance is referred to as the *hydrostatic balance.* In the r-direction the radial pressure gradient must balance the centrifugal force:

$$\rho(\Omega + \omega)^2 r = \frac{\partial p}{\partial r} \tag{4.2}$$

or

$$\rho(\Omega^2 + 2\Omega\omega + \omega^2)r = \frac{\partial p}{\partial r}.$$

Let

$$p = p_0 + p',$$

where the following expression defines p_0

$$\rho\Omega^2 r = \frac{\partial p_0}{\partial r}$$

($\Omega^2 r$ serves to modify $\vec{g}$ in geophysical systems).
So

$$\rho(2\Omega\omega + \omega^2)r = \frac{\partial p'}{\partial r}. \tag{4.3}$$

In general[2],

$$\omega \ll \Omega$$

and

$$2\Omega\omega r \cong \frac{1}{\rho}\frac{\partial p'}{\partial r} \tag{4.4}$$

The left-hand side of (4.4) is the Coriolis force per unit mass. This force is merely the linearization of the centrifugal force in a rotating coordinate frame. Equation 4.4 represents what is called *geostrophic* balance, while (4.3) is referred to as *cyclostrophic* balance.

It is sometimes convenient to express (4.4) in cartesian coordinates. Figure 4.2 permits us to view the motion in the horizontal plane within the rotating system.

At point A, $dr = dx$, and $\omega r = v$. Equation 4.4 becomes

$$2\Omega v = \frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{4.5}$$

while at point B, $dr = dy$ and $\omega r = -u$, yielding

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{4.6}$$

As already mentioned, a more systematic presentation of the above will be given in Chapter 6. The supplemental reading may also be useful to those approaching these topics for the first time.

The following remarks should be kept in mind:

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[2]N.B. The quantity $\omega/2\Omega$ is known as the *Rossby number*. More generally, it represents the relative importance of nonlinear inertial terms and the Coriolis force (based on the *system* rotation).
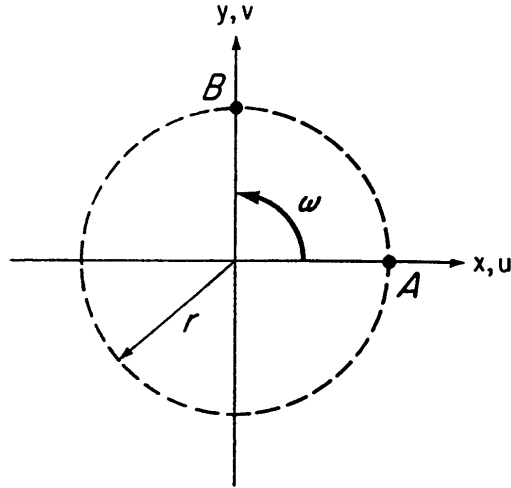
Figure 4.2: Cartesian view of circular motion.

1. Both hydrostaticity and geostrophy are static balances. It is, of course, slightly peculiar to refer to a force balance involving a moving fluid as 'static'. Nevertheless, as long as one refers to a centrifugal force, then geostrophy is a statement that two forces balance rather than a dynamic (prognostic) statement.

2. Neither hydrostaticity nor geostrophy is causal relation.

3. Strictly speaking, Equation 4.1 is true only in the absence of vertical acceleration (relative to the already rotating system) and friction. Equation 4.4 is always approximate. Nevertheless, for horizontal accelerations with time scales longer than a pendulum day (π/Ω) and vertical accelerations with time scales longer than the Brunt-Vaisala period[3] ($O(5$ minutes)), (4.1) and (4.4) remain very nearly true. These conditions apply, for the most part, to the large-scale motions of the atmosphere and oceans (at least away from frictional layers at boundaries).

4. For motions in the atmosphere and oceans the appropriate choice for Ω is its vertical component. The situation is illustrated in Figure 4.3.

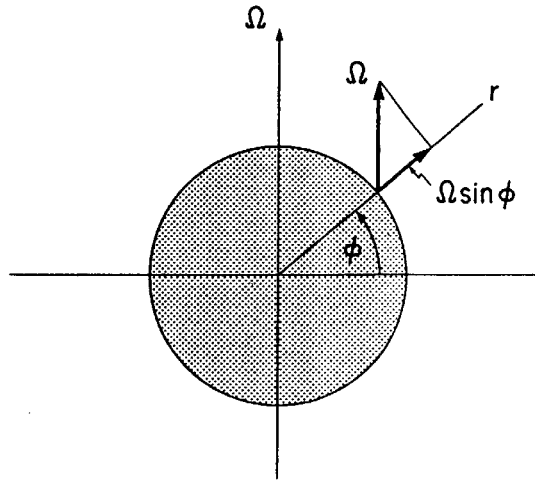[3]This is a quantity which will be defined in Chapter 6. It is a measure of stratification.

Figure 4.3: The vertical component of the rotation vector on a sphere.

In (4.4), 2Ω is replaced by $2\Omega \sin \phi$. The quantity $2\Omega \sin \phi$ is generally given the symbol f and is known as the Coriolis parameter.

5. When (4.5) and (4.6) apply, $\nabla \cdot \rho \vec{v} = 0$ (Why?). Also, pressure contours on a horizontal surface are streamlines. That is to say, geostrophic flow is *along* rather than across pressure contours. In this connection, the reader should confirm the *Law of Buys-Ballot*, namely, (in the Northern Hemisphere) when one faces in the direction of geostrophic flow, high pressure is on one's right.

The following sections contain some additional aspects of balanced flow (flow which satisfies (4.1), (4.5), and (4.6)) which will be used in Chapter 5 (on observations).

4.2 Scale height and thickness

In addition to the hydrostatic pressure relation (Equation 4.1) we have the gas law

$$p = \rho R T. \tag{4.7}$$

Using (4.7), (4.1) becomes

$$\frac{\partial p}{\partial z} = -\frac{pg}{RT} \tag{4.8}$$

or

$$\frac{\partial \ln p}{\partial z} = -\frac{1}{H} = -\frac{g}{RT}, \tag{4.9}$$

where

$$H \equiv \frac{RT}{g} = \text{local scale height}$$

$$p = p(\text{surface})e^{-x} \tag{4.10}$$

and

$$x \equiv \int_0^z \frac{dz}{H}.$$

The quantity

$$z^* = \underbrace{H_0}_{characteristic\ scale\ height} \ln\left(\frac{p_{\text{surface}}}{p}\right) \tag{4.11}$$

is almost like height if H doesn't vary 'too much'. It is also the basis of a popular coordinate system (log–p coordinates). (N.B. For a system in geostrophic and hydrostatic balance, the p-field completely determines the temperature and horizontal wind fields through Equations 4.9, 4.5, and 4.6.)

The hydrostatic relation can be rewritten

$$dp = -\rho g\, dz$$

or

$$d \ln p = -\frac{g}{RT}dz,$$

yielding

$$\int_{z_1}^{z_2} dz \equiv \text{thickness} = -\int_{p_1}^{p_2} \frac{R}{g} T \, d\ln p \qquad (4.12)$$

('thickness' refers to the vertical separation of two isobaric surfaces). The related quantity

$$\Phi \equiv \int_0^z g \, dz = -\int_{p_0}^p RT \, d\ln p \qquad (4.13)$$

is referred to as geopotential height. The quantity Φ/g is called the height field.

4.3   Thermal wind and pressure coordinates

It is easily shown[4] that

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = \left(\frac{\partial \Phi}{\partial x}\right)_p \qquad (4.14)$$

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = \left(\frac{\partial \Phi}{\partial y}\right)_p . \qquad (4.15)$$

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[4]

$$p = p(x, y, z, t)$$

and

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz + \frac{\partial p}{\partial t}dt = 0,$$

which implies

$$\frac{\partial z}{\partial x} = -\left(\frac{\partial p}{\partial x}\right)_z / \left(\frac{\partial p}{\partial z}\right)_x = \frac{1}{\rho g}\frac{\partial p}{\partial x},$$

which in turn implies

$$\left(\frac{\partial \Phi}{\partial x}\right)_p = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z .$$

Equations 4.5 and 4.6 then become

$$fu \;=\; -\frac{\partial \Phi}{\partial y} \tag{4.16}$$

$$fv \;=\; +\frac{\partial \Phi}{\partial x}. \tag{4.17}$$

Now from (4.13), $\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$. So differentiating (4.16) and (4.17) with respect to p yields

$$p\frac{\partial v}{\partial p} \;=\; -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p = \frac{\partial v}{\partial \ln p} \tag{4.18}$$

$$p\frac{\partial u}{\partial p} \;=\; +\frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p = \frac{\partial u}{\partial \ln p}. \tag{4.19}$$

Equations 4.18 and 4.19 may be rewritten to yield the *thermal wind* relations (in log–p coordinates):

$$\frac{1}{H}\frac{\partial u}{\partial \ln p} \;=\; -\frac{H_0}{H}\frac{\partial u}{\partial z^*} = +\frac{g}{fT}\left(\frac{\partial T}{\partial y}\right)_p \tag{4.20}$$

$$\frac{1}{H}\frac{\partial v}{\partial \ln p} \;=\; -\frac{H_0}{H}\frac{\partial v}{\partial z^*} = -\frac{g}{fT}\left(\frac{\partial T}{\partial x}\right)_p. \tag{4.21}$$

Under many circumstances $\frac{H_0}{H} \sim 1$, $z^* \sim z$, and pressure surfaces are far more horizontal than temperature surfaces. Then

$$\frac{\partial u}{\partial z} \;\approx\; -\frac{g}{fT}\frac{\partial T}{\partial y} \tag{4.22}$$

$$\frac{\partial v}{\partial z} \;\approx\; +\frac{g}{fT}\frac{\partial T}{\partial x}. \tag{4.23}$$

From the above expressions for the *thermal wind* we see that the counterpart of the Law of Buys-Ballot for shear is simply that if one is facing in the direction in which wind is increasing upward, high temperature is on one's right (again for the Northern Hemisphere).