Chapter 11

Rossby waves

Supplemental reading:

Pedlosky[1] (1979), sections 3.1–3

11.1 Shallow water equations

When considering the general problem of linearized oscillations in a static, arbitrarily stratified atmosphere (or ocean for that matter), we find that there is separability of latitude and altitude dependence (in spherical coordinates). For simplicity in what follows we will consider a β plane instead of a spherical surface. As on a sphere, our equations naturally divide into two sets.

The first set consists in the following three equations:

$$i\sigma u'_n - f v'_n = ik\Phi'_n \tag{11.1}$$

$$i\sigma v'_n + f u'_n = -\frac{\partial \Phi'_n}{\partial y} \tag{11.2}$$

and

$$iku'_n + \frac{\partial v'_n}{\partial y} = -\frac{i\sigma}{gh_n}\Phi'_n. \tag{11.3}$$

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[1]Pedlosky offers very detailed explorations of the topics we are skimming. Don't get too discouraged in reading this book. It was the basis for a five trimester course at the University of Chicago

The second set consists in the following two equations:

$$-\frac{i\sigma}{gh_n}\Phi'_n + \frac{dw_n^*}{dz^*} - w_n^* = 0 \tag{11.4}$$

and

$$i\sigma\frac{d\Phi'_n}{dz^*} + w_n^* R\left(\frac{dT_0}{dz^*} + \frac{RT_0}{c_p}\right) = \kappa J_n. \tag{11.5}$$

The first three equations can be combined to form Laplace's Tidal Equation (for a sphere) or its equivalent on the β plane.

The second set determines vertical structure – leading to a *vertical structure equation*

$$\frac{d^2\tilde{w}_n}{dz^{*2}} + \left\{\frac{1}{h_n}\left(\frac{dH}{dz^*} + \kappa H\right) - \frac{1}{4}\right\}\tilde{w}_n = \frac{\kappa J_n}{gh_n}e^{-z^*/2}, \tag{11.6}$$

where

$$w_n^* = e^{z^*/2}\tilde{w}_n. \tag{11.7}$$

At a horizontal rigid lower boundary, the boundary condition on $\tilde{w}_n$ is

$$\frac{d\tilde{w}_n}{dz^*} + \left(\frac{H}{h_n} - \frac{1}{2}\right)\tilde{w}_n = 0 \text{ at } z^* = 0. \tag{11.8}$$

At a free upper surface we would require

$$\tilde{w}_n = 0 \text{ at } z^* = H^*. \tag{11.9}$$

For an unbounded atmosphere we require boundedness and/or the radiation condition.

In the case of forced oscillations, the forcing determines the frequency and zonal wavenumber, Laplace's Tidal Equation defines and eigenfunction-eigenvalue problem where the eigenfunctions are known as Hough Functions and the eigenvalues are the equivalent depths associated with different meridional structures. The forcing, J, is then expanded in terms of these Hough Functions, and the vertical structure is used to solve for the vertical structure pertaining to each Hough mode.

For free oscillations, on the other hand, we solve (11.6) subject to the boundary conditions for $J = 0$. This yields the equivalent depths of the
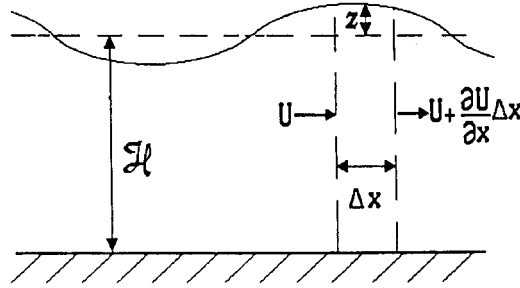
Figure 11.1: Shallow water geometry.

atmosphere (or ocean). For these depths we then solve (11.1)–(11.3) where frequency rather than h is the eigenvalue. The resulting modes are known as free oscillations.

It must be emphasized that the eigenfunctions of (11.6) do not, in general, form a complete set. For example, for a semi-infinite atmosphere with an isothermal basic state there exists a single equivalent depth for the atmosphere. This is sometimes referred to as the barotropic mode. However, the so called baroclinic modes are usually artifacts resulting from the artificial presence of a reflecting lid. In general,

$$h = \gamma H. \tag{11.10}$$

We have also shown that for a shallow fluid with a free surface and $N^2 = 0$ (i.e., $\frac{dH}{dz^*} + \kappa H \equiv 0$) there is also a single equivalent depth

$$h = \mathcal{H}^* H = \mathcal{H} = \text{depth of unperturbed fluid.} \tag{11.11}$$

Under the latter circumstances, (11.1)–(11.3) are known as the *shallow water equations*. The point is that for a shallow fluid u' and v' are essentially independent of z, and Φ is just $g\times$ surface displacement. Equation 11.3 can then be directly interpreted as the continuity equation. This is readily seen if we ignore y-variation. Referring to Figure 11.1 we see that as $\Delta x \rightarrow$

0, $\mathcal{H}\frac{\partial u'}{\partial x} + \frac{\partial z'}{\partial t} = 0$. Under these circumstances there is no further need for the vertical structure equation. What our analysis shows is that the shallow water equations are always appropriate to hydrostatic oscillations on a static basic state – subject to the reinterpretation of h. Our analysis, therefore, serves to justify and generalize the use of the shallow water equations.

The shallow water equations are used as model equations in both meteorology and oceanography. In most applications they are not considered approximations. We shall make extensive use of these equations in the following two subsections where we investigate the effects of f, the Coriolis parameter, varying with y.

It should be remembered, however, that once we obtain a relation among frequency, horizontal wavenumbers, and the depth of the 'shallow water', we are always free to reinterpret the depth as an equivalent depth – and to solve for the depth in terms of the other parameters in order to apply the results to problems of forced rather than free oscillations.

11.2 Rossby waves

When f is constant and (11.1)–(11.3) are applied to a channel geometry, they describe easterly and westerly gravity waves modified by rotation. We will now allow f to vary in the same way that density varies in a Boussinesq fluid; that is, we will take f to be constant unless it is differentiated – in which case we will take $\frac{df}{dy} = \beta = $ constant. This is known as the β-plane approximation and is, in fact, due to Rossby. Rossby intuitively realized that for large-scale meteorological systems the first modification on a sphere to the f-plane equations arises from the variation of f with latitude. Subsequently, scaling analysis has been used to more rigorously justify this approximation.

Our starting point will be (11.1)–(11.3). With variable f, it will prove more convenient to reduce (11.1)-(11.3) to a single equation in v' rather than Φ'. For free oscillations it doesn't matter which field we solve for. Eliminating u' between (11.1) and (11.2) we get

$$(f^2 - \sigma^2)v' = -i\sigma\frac{\partial \Phi'}{\partial y} + ikf\Phi'. \tag{11.12}$$

Similarly, eliminating u' between (11.1) and (11.3) yields:

$$i\sigma\frac{\partial v'}{\partial y} + ikfv' = \frac{\sigma^2}{gh}\Phi' - k^2\Phi'. \tag{11.13}$$

We next operate on (11.13) with $\left(-i\sigma\frac{\partial}{\partial y} + ikf\right)$ to get

$$\left(-i\sigma\frac{\partial}{\partial y} + ikf\right)\left(i\sigma\frac{\partial}{\partial y} + ikf\right)v' = \left(\frac{\sigma^2}{gh} - k^2\right)\left(-i\sigma\frac{\partial}{\partial y} + ikf\right)\Phi',$$

and using (11.12) we get

$$\left(-i\sigma\frac{\partial}{\partial y} + ikf\right)\left(i\sigma\frac{\partial}{\partial y} + ikf\right)v' = \left(\frac{\sigma^2}{gh} - k^2\right)\left(f^2 - \sigma^2\right)v',$$

or

$$\left\{\sigma^2\frac{\partial^2}{\partial y^2} \underbrace{+\sigma kf\frac{\partial}{\partial y}}_{1} +\sigma k\beta \underbrace{-\sigma kf\frac{\partial}{\partial y}}_{1} -\underbrace{k^2f^2}_{2}\right\}v'$$

$$= \left\{\frac{\sigma^2 f^2}{gh} - \frac{\sigma^4}{gh} -\underbrace{k^2 f^2}_{2} +k^2\sigma^2\right\}v',$$

where the terms with numbered underbraces cancel. Finally, (dividing by σ^2) we get

$$\frac{\partial^2 v'}{\partial y^2} + \left(\frac{k\beta}{\sigma} + \frac{\sigma^2 - f^2}{gh} - k^2\right)v' = 0. \tag{11.14}$$

(It turns out that we also have Kelvin wave solutions for which $v' \equiv 0$. These are described at the end of this chapter.) If we again adopt a channel geometry where

$$v' = 0 \text{ at } y = 0, d, \tag{11.15}$$

then (11.14) has solutions

$$v' = \sin\frac{n\pi y}{d},$$

and (11.14) becomes

$$-\left(\frac{n\pi}{d}\right)^2 + \left(\frac{k\beta}{\sigma} + \frac{\sigma^2 - f^2}{gh} - k^2\right) = 0. \tag{11.16}$$

Recall that for free oscillations, h is given, and (11.16) is solved for σ, whereas in a forced problem, σ and k are given and (11.16) is solved for h. Although we shall return to free oscillations shortly, some useful insights can be gotten by looking at (11.16) as an equation for h:

$$gh = \frac{\sigma^2 - f^2}{(\ell^2 + k^2) - \frac{k\beta}{\sigma}} = \frac{\sigma^2 - f^2}{k^2\{\frac{\ell^2}{k^2} + 1 - \frac{\beta}{\sigma k}\}}, \tag{11.17}$$

where $\ell = \frac{n\pi}{d}$.

In the absence of β, oscillations for which $\sigma^2 < f^2$ cannot propagate vertically (propagation requires $0 < h \tilde{<} H$; for $\sigma^2 < f^2$, h is negative). When β is included, (11.17) shows that this is no longer strictly true. For easterly waves (where σ and k have the same sign) and sufficiently *small* σ, vertical propagation is again possible. The resulting waves are called internal Rossby waves. Some quantitative estimates will help us determine the relevant scales for such waves. Let us take our β-plane to be centered at $45°$ latitude. Then

$$f = 2\Omega \sin\phi \approx 10^{-4} \sec^{-1}$$

$$\beta = \frac{df}{dy} = \frac{2\Omega \cos\phi}{a} \approx \frac{10^{-4} \sec^{-1}}{a},$$

where $a =$ Earth's radius ≈ 6400 km.

For convenience let's take $n = 1$ and $d = a$. Then

$$\ell^2 \approx \left(\frac{\pi}{a}\right)^2.$$

Also

$$k = \frac{s}{a \cos\phi} = \frac{s\sqrt{2}}{a}$$

and

$$k^2 = \frac{2s^2}{a^2}.$$

From (11.17) we see that positive equivalent depths are possible for

$$0 < \sigma < \frac{\beta/k}{1 + \frac{\ell^2}{k^2}} = \frac{f/\sqrt{2}s}{1 + \frac{1}{2}(\frac{\pi}{s})^2} \leq \frac{f}{2\pi}. \tag{11.18}$$

(The reader should derive the last inequality.) In other words, Rossby waves are associated with periods for which $\sigma^2 \ll f^2$.

11.2.1 Planetary scale internal stationary waves

There is a quick application of the above. Just as flow over individual mountains is a major forcing of internal gravity waves, so too is flow over larger-scale surface features (the Tibetan Plateau, for example) a major forcing of internal Rossby waves. Now as we saw in Chapter 10 the inclusion of a basic flow can substantially complicate matters (indeed, if U_0 varies with z then, at least on a rotating sphere, we lose separability). However, in the trivial instance where $U_0 = $ constant, the primary effect is only to replace σ with $\sigma + kU_0$. (The effect on the lower boundary condition can be more complicated – but we will ignore this.) Now for stationary waves $\sigma = 0$ and (11.18) becomes

$$0 < kU_0 < \frac{f/\sqrt{2}s}{1 + \frac{1}{2}(\frac{\pi}{s})^2}$$

or

$$0 < U_0 < \frac{fa}{2s^2(1 + \frac{1}{2}(\frac{\pi}{s})^2)} = \frac{fa/2}{s^2 + \frac{\pi^2}{2}} \equiv U_{trap}, \tag{11.19}$$

where

$$U_{trap} = \begin{array}{l} 54\text{m/s for } s = 1 \\ 36\text{m/s for } s = 2 \\ 23\text{m/s for } s = 3, \end{array}$$

Only sufficiently weak westerlies permit stationary wave propagation. We see from (11.19) that for tropospheric winds stationary wavenumbers three and greater will not readily propagate into the stratosphere thus accounting for the predominance of wavenumbers one and two in the winter stratosphere. We also see that the summer stratospheric easterlies will effectively block the propagation of all stationary waves accounting for the observed zonal

character of the summer stratospheric circulation. These results were first noted by Charney and Drazin (1961).

11.2.2   Free oscillations

Let us now return to free oscillations. Rewriting (11.16) we get

$$\frac{k\beta}{\sigma} + \frac{\sigma^2 - f^2}{gh} - \left(k^2 + \left(\frac{n\pi}{d} \right)^2 \right) = 0. \tag{11.20}$$

Equation 11.20 is cubic in σ. We may anticipate that the three roots will correspond to two gravity waves and a Rossby wave. In general, gravity wave frequencies (for positive h) will exceed f so the term $\frac{k\beta}{\sigma}$ will be negligible. Similarly, for Rossby waves $\sigma^2 \ll f^2$. Thus (11.20) has the following approximate solutions:

$$\sigma^2 \approx gh \left(k^2 + \left(\frac{n\pi}{d} \right)^2 \right) + f^2 \tag{11.21}$$

and

$$\frac{k\beta}{\sigma} = \frac{f^2}{gh} + k^2 + \left(\frac{n\pi}{d} \right)^2$$

or

$$\sigma = \frac{k\beta}{k^2 + (\frac{n\pi}{d})^2 + \frac{f^2}{gh}}. \tag{11.22}$$

In contrast to gravity waves, which can have easterly and westerly phase speeds, Rossby waves always have easterly phase speed (relative to U_0). East–west asymmetry is always a characteristic of motions for which β is important.

Several properties of (11.22) are worth noting:

(i) σ has a maximum value for a particular value of k (which you will work out as an exercise). Pedlosky (1979) has an elegant explanation of this which we shall go over as soon as we develop some theorems on vorticity conservation.

(ii) With a constant zonal flow we can replace σ with $\sigma + kU_0$. Also $c = -\frac{\sigma}{k}$. (11.22) becomes

$$c = U_0 - \frac{\beta}{k^2 + (\frac{n\pi}{d})^2 + \frac{f^2}{gh}}. \tag{11.23}$$

Note that as $(k^2 + (\frac{n\pi}{d})^2)$ gets larger the Rossby wave moves more and more nearly with the basic flow U_0. This is completely consistent with observations.

(iii) The velocity fields associated with Rossby waves are almost in geostrophic balance. Nonetheless, the outright assumption of geostrophy would not permit us to evaluate the time evolution of Rossby waves. The development of an appropriate approximate set of equations which exploit geostrophy but still describe Rossby waves will be one of our tasks.

(iv) The dispersive properties of Rossby waves clearly suggests that it is Rossby waves and not gravity waves which describe large-scale meteorological systems.

11.3   Remark on Kelvin waves: The case of $v' \equiv 0$.

The solutions in Section 2 assume $v' \neq 0$. We here consider the situation where $v' \equiv 0$. There exists a solution in this case known as a Kelvin wave. As an exercise you will derive the properties of Kelvin waves on what is known as an equatorial β -plane (where $f = \beta y$). Here we will take $f = f_0$. Equations 11.1–11.3 become

$$i\sigma u' = -ik\Phi' \tag{11.24}$$

$$fu' = -\frac{\partial \Phi'}{\partial y} \tag{11.25}$$

and

$$iku' = -\frac{i\sigma}{gH}\Phi'. \tag{11.26}$$

From (11.24) and (11.26) we get

$$\frac{\sigma^2}{k^2} = gH, \tag{11.27}$$

while from (11.24) and (11.25) we get

$$\frac{\partial \Phi'}{\partial y} - f_0 \frac{k}{\sigma} \Phi' = 0. \tag{11.28}$$

The solution to (11.28) is

$$\Phi' = Ce^{f_0(\frac{k}{\sigma})y}.$$

Note that for $k\sigma < 0$ (i.e., westerly or eastward propagating waves) amplitudes decay away from the boundary at $y = 0$, while for $k/\sigma > 0$ (i.e., easterly or westward propagating waves) amplitudes decay away from the boundary at $y = d$. In a closed basin, Kelvin waves travel around the boundary of the basin in a counterclockwise direction (when $f_0 > 0$). Kelvin waves satisfy the dispersion relation for a gravity wave in the absence of rotation; at the same time, the Kelvin wave's horizontal velocity field is in geostophic balance.