Objective analysis [scalar fields]

We would like to estimate a property $S(\mathbf{x})$ at a point $\mathbf{x}$ given a set of observations $s(\mathbf{x}_i)$ (containing errors) of the property at other spatial points. In the absence of other information (climatology, for example), we assume that the data mean represents the true mean and use a linear estimator for the deviations

$$\tilde{S}'(\mathbf{x}) = \sum s'(\mathbf{x}_i)a(\mathbf{x}_i, \mathbf{x})$$

or, using summation convention,

$$\tilde{S}'(\mathbf{x}) = s'(\mathbf{x}_i)a(\mathbf{x}_i, \mathbf{x}) \tag{1}$$

The problem now becomes the choice of a.

We form an error estimate

$$\begin{aligned}\epsilon &= \frac{1}{2}\langle[\tilde{S}'(\mathbf{x}) - S'(\mathbf{x})]^2\rangle \\ &= \frac{1}{2}\langle s'(\mathbf{x}_i)s'(\mathbf{x}_j)\rangle a(\mathbf{x}_i, \mathbf{x})a(\mathbf{x}_j, \mathbf{x}) - \langle S'(\mathbf{x})s'(\mathbf{x}_i)\rangle a(\mathbf{x}_i, \mathbf{x}) + \frac{1}{2}\langle S'(\mathbf{x})S'(\mathbf{x})\rangle \end{aligned} \tag{2}$$

We seek the minimum error with respect to the values of the coefficients $a(\mathbf{x}_i, \mathbf{x})$

$$\frac{\partial \epsilon}{\partial a(\mathbf{x}_i, \mathbf{x})} = 0$$

which implies

$$\langle s'(\mathbf{x}_i)s'(\mathbf{x}_j)\rangle a(\mathbf{x}_j, \mathbf{x}) = \langle S'(\mathbf{x})s'(\mathbf{x}_i)\rangle \tag{3}$$

The symmetry of $\langle s'(\mathbf{x}_i)s'(\mathbf{x}_j)\rangle$ has been used. We write this in terms of the covariance for the field

$$C(\mathbf{x} - \mathbf{x}') = \langle S'(\mathbf{x})S'(\mathbf{x}')\rangle$$

assuming that the measurement noise is uncorrelated and has variance σ^2

$$[C(\mathbf{x}_i - \mathbf{x}_j) + \sigma^2\delta_{ij}]a(\mathbf{x}_j, \mathbf{x}) = C(\mathbf{x}_i - \mathbf{x})$$

If we know or can approximate the covariance function, we can set up and solve this linear system to give $a(\mathbf{x}_i, \mathbf{x})$ for any target point $\mathbf{x}$

$$a(\mathbf{x}_j, \mathbf{x}) = [C(\mathbf{x}_i - \mathbf{x}_j) + \sigma^2\delta_{ij}]^{-1}C(\mathbf{x}_i - \mathbf{x}) \tag{4}$$

Not only can we substitute this in (1) to find the estimated field at $\mathbf{x}$, we can also get an estimate of the error by using (3) and (4) in (2)

$$\epsilon = \frac{1}{2}C(0) - \frac{1}{2}C(\mathbf{x} - \mathbf{x}_j)[C(\mathbf{x}_i - \mathbf{x}_j) + \sigma^2\delta_{ij}]^{-1}C(\mathbf{x}_i - \mathbf{x})$$

Note that the errors depend only on the sampling positions and C, σ. Therefore we can design sampling strategies given an estimate of the covariance and the noise.

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