15. Higher-order balance systems

The quasi-geostrophic system is adequate for very low Rossby number and Froude number flows, such as characterize the ocean even at the "mesoscale" (O(100 km)). The low Rossby-Froude number approximation in the atmosphere is not as good, and the quasi-geostrophic equations do not work as well, *quantitatively.* Higher-order balance approximations are needed for an accurate diagnosis of quasi-balanced dynamical processes in the atmosphere, or for numerical weather prediction, if we were to use a quasi-balanced system. (For various reasons, balanced equations have been abandoned as a basis for NWP and the full set of "primitive," or hydrostatic, equations are used.)

One basis of higher-order balance systems involves approximating the *horizontal divergence as small compared to the vertical component of vorticity* in a fluid flow. This approximation also leads to a system in which the velocity field is instantaneously related to the mass distribution.

Begin with the hydrostatic approximation to the momentum equations in pressure coordinates:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \omega \frac{\partial \mathbf{V}}{\partial p} = -\nabla \varphi - f \hat{k} \times \mathbf{V} + \mathbf{F}. \tag{15.1}$$

Now operate on the above with the horizontal divergence operator, $\boldsymbol{\nabla}_2$. The result may be written

$$\frac{dD}{dt} = -\nabla^2 \varphi + (f + \zeta)\zeta - \beta u - \frac{\partial \omega}{\partial x}\frac{\partial u}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial v}{\partial p} - S^2 + \boldsymbol{\nabla}_2 \cdot \mathbf{F}, \tag{15.2}$$

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where D is the horizontal divergence, $\boldsymbol{\nabla}_2 \cdot \mathbf{V}$, ζ is the vertical component of vorticity, and

$$S^2 \equiv \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2, \tag{15.3}$$
$$\beta \equiv \frac{df}{dy}.$$

If we assume that

$$\left|\frac{dD}{dt}\right| \ll |(f + \zeta)\zeta|, \tag{15.4}$$

then (15.2 may be approximated by the *nonlinear balance equation:*

$$\nabla^2 \varphi = (f + \zeta)\zeta - \beta u - S^2 - \frac{\partial \omega}{\partial x}\frac{\partial u}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial v}{\partial p} + \boldsymbol{\nabla}_2 \cdot \mathbf{F}. \tag{15.5}$$

Unlike the geostrophic relations, (15.5) represents the geopotential field as a *nonlinear* function of the velocity distribution. Note that we have assumed that the total derivative of D is small compared to *all* the terms in (15.5).

At the same time, the hydrostatic approximation to Ertel's potential vorticity is

$$q = -g\left[(f + \zeta)\frac{\partial \theta}{\partial p} - \frac{\partial v}{\partial p}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial p}\frac{\partial \theta}{\partial y}\right], \tag{15.6}$$

and conservation of potential vorticity is governed by (cf. 5.10)

$$\frac{\partial q}{\partial t} = -\mathbf{V} \cdot \boldsymbol{\nabla} q - \omega \frac{\partial q}{\partial p} + \alpha(\nabla \times \mathbf{F}) \cdot \boldsymbol{\nabla}\theta + \alpha(2\boldsymbol{\Omega} + \boldsymbol{\nabla} \times \mathbf{V}) \cdot \boldsymbol{\nabla}\frac{d\theta}{dt}. \tag{15.7}$$

Also, remember that θ is related hydrostatically to φ via (8.33), whose atmosphere part we write

$$\pi \frac{\partial \varphi}{\partial p} = -\theta, \tag{15.8}$$

where

$$\pi \equiv p \left(\frac{p_0}{p} \right)^{R/c_p} /R. \tag{15.9}$$

To this we add the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0. \tag{15.10}$$

Clearly, the five equations (15.5)–(15.8) and (15.10) contain the six dependent variables u, v, ω, θ, φ, and q and so do not constitute a closed system. Progress can be made by ordering the flow according to the relative magnitudes of its irrotational and nondivergent components. We first express the horizontal flow components in terms of a mass streamfunction, ψ, and a velocity potential, χ:

$$\begin{aligned} u &= -\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x}, \\ v &= \frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y}, \end{aligned} \tag{15.11}$$

where, by (15.10),

$$\nabla^2 \chi = -\frac{\partial \omega}{\partial p}. \tag{15.12}$$

Using (15.11) and (15.8), the set (15.5)–(15.7) can be written

$$\begin{aligned} \nabla^2 \varphi = (f + \nabla^2 \psi)\nabla^2 \psi + \beta \left(\frac{\partial \psi}{\partial y} - \frac{\partial \chi}{\partial x} \right) - S^2(\psi, \chi) \\ - \frac{\partial \omega}{\partial x}\frac{\partial u}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial v}{\partial p} + \boldsymbol{\nabla}_2 \cdot \mathbf{F}, \end{aligned} \tag{15.13}$$

$$\begin{aligned} q = g \Bigg[(f + \nabla^2 \psi)\frac{\partial}{\partial p} \left(\pi \frac{\partial \varphi}{\partial p} \right) - \pi \frac{\partial^2 \varphi}{\partial p \partial x}\frac{\partial}{\partial p} \left(\frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y} \right) \\ + \pi \frac{\partial^2 \varphi}{\partial y \partial p}\frac{\partial}{\partial p} \left(\frac{\partial \chi}{\partial x} - \frac{\partial \psi}{\partial y} \right) \Bigg], \end{aligned} \tag{15.14}$$

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$$\frac{\partial q}{\partial t} = \left(\frac{\partial \psi}{\partial y} - \frac{\partial \chi}{\partial x} \right) \frac{\partial q}{\partial x} - \left(\frac{\partial \psi}{\partial x} - \frac{\partial \chi}{\partial y} \right) \frac{\partial q}{\partial y} - \omega \frac{\partial q}{\partial p} \qquad (15.15)$$

$$+ \text{ friction and heating.}$$

Consistent with the approximation given by (15.4), we assume that

$$|\chi| < |\psi| . \qquad (15.16)$$

Consider an iterative process in which, in the first step, we take χ (and therefore ω) to be zero in (15.13)–(15.15). In that case, we have a closed system in the variables q, φ, and ψ, and, given appropriate boundary conditions, *we can step (15.15) forward one time step, then invert the system (15.13)–(15.14) to get ψ and φ at the next time step.* This gives us all the fields q, ψ, and φ at the beginning *and* end of a time step. Now, we introduce another equation, the thermodynamic equation, which can be written (backwards) in terms of φ:

$$\omega \frac{\partial}{\partial p} \left(\pi \frac{\partial \varphi}{\partial p} \right) = -\pi \left(-\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x} \right) \frac{\partial^2 \varphi}{\partial p \partial x} - \pi \left(\frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y} \right) \frac{\partial^2 \varphi}{\partial p \partial y}$$
$$\frac{-\pi}{\Delta t} \frac{\partial \varphi}{\partial p} \bigg|_t^{t+\Delta t} - Q, \qquad (15.17)$$

where Δt is the time step. Again taking χ to be zero on the right side of (15.17) we can now solve (15.17) for ω since we know φ and ψ at the beginning and end of the time step. We can then solve (15.12) for an estimate of χ, and start the whole time step over again, this time with a nonzero estimate for χ. This gives us new estimates of q, ψ, and φ at the end of the time step and therefore, from (15.17), new estimate of ω and χ. Although no formal proof has been developed that shows

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that the iteration converges, it seems to in practice. Experience shows that this method of solution of the equations describes atmospheric motions very well.

The ability of a system like this one to describe quite accurately atmospheric and oceanic flows raises an important philosophical question: Can general flows be separated into quasi-balanced parts, uniquely associated with the potential vorticity distribution, and everything that is left over after the balanced part is accounted for (e.g., inertia-gravity waves, three-dimensional turbulence)? The precise answer seems to be "no"; there will always be some interaction between the quasi-balanced and unbalanced flow components that render their separation imprecise. Nonetheless, these interactions are *usually* (but not always) weak in geophysical fluid flows, and so it is useful to talk about these components separately. Our *working* definition of quasi-balanced flow is *flow uniquely associated with and calculable from the potential vorticity distribution.*

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