6. Internal waves

Consider a continuously stratified fluid with $\rho_o(z)$ the vertical density profile.
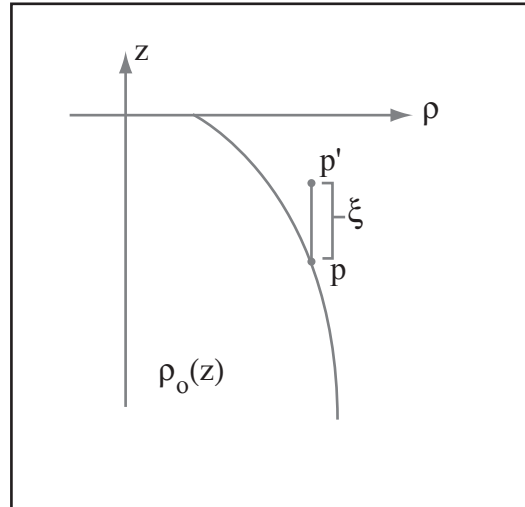

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Figure 1.

At a point P raise a parcel of water by small amount ξ from its equilibrium position P adiabatically. The change in pressure experienced by the parcel is

$$dp = -\rho_o\, g\xi$$

if $\rho_o(z)$ is the medium density, while the change in density is $d\rho = \dfrac{dp}{c_s^2}$ [Remember the

definition of adiabatic compressibility $\left(\dfrac{\partial p}{\partial \rho}\right) = c_s^2$]. The buoyancy force acting on the

particle induces an acceleration because the density of the particle at the displaced

position is different from the background density of the medium

$$\rho_o\,|_{P'} = \rho_o\,|_P + \frac{d\rho_o}{dz}\,\xi \quad \text{and}$$

$$\rho\,|_{\text{particle at P'}} = \rho\,|_{\text{particle at P}+d\rho} = \rho_o\frac{-\rho_o g\xi}{c_s^2}$$

1

$$\rho_o \frac{d^2\xi}{dt^2} = g[(\rho_o + \rho_{oz}\xi) - (\rho_o - \frac{\rho_o g \xi}{c_s^2})]$$

$$\frac{d^2\xi}{dt^2} = \frac{g}{\rho_o}[(\rho_{oz} + \frac{\rho_o g}{c_s^2})\xi]$$

$$\frac{d^2\xi}{dt^2} + (-\frac{g\rho_{oz}}{\rho_o} - \frac{g^2}{c_s^2})\xi = 0$$

Define $\quad N^2(z) = \frac{-g}{\rho_o}\frac{d\rho_o}{dz} - \frac{g^2}{c_s^2} \quad$ the Brünt-Vaisala frequency or buoyancy frequency

$$\frac{d^2\xi^2}{dt^2} + N^2\xi = 0$$

is the equation of the harmonic oscillator with solution $e^{\pm iNt}$.

Thus the parcel oscillates about its equilibrium position and the Brünt-Vaisala frequency is the natural frequency of oscillation determined by the local density stratification and the fluid's compressibility. As we saw, in the ocean compressibility effects are negligible and we can assume

$$N^2 = \frac{-g}{\rho_o}\frac{\partial \rho_o}{\partial z}$$

The motion is the oscillation of the particle around the equilibrium position with frequency N. Let us compare the frequency of surface gravity waves with the buoyancy frequency and use deep water gravity waves

$$\frac{\omega_{int.}}{\omega_{surf.}} = \frac{N}{\sqrt{gk}} = (-\frac{1}{k\rho_o}\frac{\partial \rho_o}{\partial z})^{1/2} \simeq (-\frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z}\lambda)^{1/2} \approx o(\frac{\Delta\rho_o}{\rho_o})^{1/2}$$

$$k \sim \frac{1}{\lambda}; \quad \lambda\frac{\partial \rho_o}{\partial z} = \Delta\rho_o \qquad \text{density change over } \lambda$$

$\Delta\rho_o \sim 10^{-3}$ over total depth:

$\omega_{int} \ll \omega_{surface}$.

The frequency of internal waves is much less than the frequency of surface gravity waves. The restoring force for surface gravity waves is g; the restoring force for internal waves is $\dfrac{\Delta\rho_o}{\rho_o} g$, reduced gravity.

Internal waves in an unbounded fluid

Consider an incompressible, stratified non-rotating fluid characterized by a basic state of rest and hydrostatic equilibrium:

$$u \equiv o, \quad \rho = \rho_o(z), \quad p = p_o(z) \quad \text{and} \quad \frac{dp_o}{dz} = -g\rho_o$$

The fluid experiences small amplitude perturbations around the basic state, so we can use the linearized equations of motion, assuming also that the motion is frictionless and adiabatic:

(1) $\rho_o \dfrac{\partial u}{\partial t} = -\dfrac{\partial p}{\partial x}$ $\qquad\qquad$ $\rho_{total} = \rho_o + \rho; \quad \rho \ll \rho_o$

(2) $\rho_o \dfrac{\partial v}{\partial t} = -\dfrac{\partial p}{\partial y}$ $\qquad\qquad$ $p_{total} = p_o + p; \quad p \ll p_o$

(3) $\rho_o \dfrac{\partial w}{\partial t} = -\dfrac{\partial p}{\partial z} - \rho g$ $\qquad\qquad$ ρ = perturbation density

(4) $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$ $\qquad\qquad$ p = perturbation pressure

(5) $\dfrac{\partial p}{\partial t} + w\dfrac{\partial \rho_o}{\partial z} = 0$ $\qquad\qquad$ The basic reference state cancels out in 3

If $u = v = o$ and $p = o$, we have the special solution

$$\rho_o \frac{\partial w}{\partial t} = -\rho g; \quad \frac{\partial \rho}{\partial t} = -w \frac{\partial \rho_o}{\partial z}$$

$$\rho_o \frac{\partial^2 w}{\partial t^2} = -g \frac{\partial \rho}{\partial t} = +\frac{g \partial \rho_o}{\partial z} w \rightarrow \frac{\partial^2 w}{\partial t^2} + \left(\frac{-g}{\rho_o} \frac{\partial \rho_o}{\partial z}\right) w = o$$

General equation

Differentiate (4): $\quad \dfrac{\partial}{\partial t} \dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial t} \dfrac{\partial v}{\partial y} + \dfrac{\partial}{\partial t} \dfrac{\partial w}{\partial z} = 0$

Substitute into it $\left(\dfrac{\partial u}{\partial t}, \dfrac{\partial v}{\partial t}\right)$ given by (1) and (2)

$$\frac{\partial}{\partial x}\left(-\frac{1}{\rho_o} \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{1}{\rho_o} \frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial t}\right) = 0$$

(I) $\qquad \dfrac{\partial^2 w}{\partial z \partial t} = \dfrac{1}{\rho_o} \nabla_H^2 p \quad$ where $\nabla_H^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

Eliminate ρ between (3) and (5)

$$\rho_o \frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2 p}{\partial z \partial t} - g \frac{\partial \rho}{\partial t} = -\frac{\partial^2 p}{\partial z \partial t} + gw \frac{d\rho_o}{\partial z}$$

or $\qquad \dfrac{\partial^2 w}{\partial t^2} + \left(-\dfrac{g}{\rho_o} \dfrac{\partial \rho_o}{\partial z}\right) w = -\dfrac{1}{\rho_o} \dfrac{\partial^2 p}{\partial z \partial t}$

(II) $\qquad \dfrac{\partial^2 w}{\partial t^2} + N^2 w = -\dfrac{1}{\rho_o} \dfrac{\partial^2 p}{\partial z \partial t}$

Eliminate p between (I) and (II) taking ∇_H^2 of (II)

$$\nabla_H^2 \frac{\partial^2}{\partial t^2} \nabla_H^2 w + N^2 \nabla_H^2 w = -\frac{1}{\rho_o} \frac{\partial^2}{\partial z \partial t^2} \nabla_H^2 p = -\frac{1}{\rho_o} \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial z}\left(\rho_o \frac{\partial w}{\partial z}\right)$$

Take to the LHS and factorize $\dfrac{\partial^2}{\partial t^2}$

Internal wave equation
$$\frac{\partial^2}{\partial t^2}\left[\nabla_H^2 w + \frac{1}{\rho_o}\frac{\partial}{\partial z}(\rho_o\frac{\partial w}{\partial z})\right] + N^2\nabla_H^2 w = 0$$

Consider the last term in square bracket:

$$\frac{1}{\rho_o}\frac{\partial}{\partial z}(\rho_o\frac{\partial w}{\partial z}) = \frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z}\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial z^2}$$

Ratio

$$\frac{\frac{\partial \rho_o}{\partial z}\frac{\partial w}{\partial z}}{\rho_o\frac{\partial^2 w}{\partial z^2}} = \frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z}\bullet\frac{1}{d}d^2 = \frac{d}{\rho_o}\frac{\partial \rho_o}{\partial z} << 1 = \frac{d}{D}$$

d = vertical scale of w ~ thickness of thermocline

and D is the ocean depth or atmospheric height -> valid also in the atmosphere. So we

can ignore the first term and we simplify to:

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]w + N^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)w = 0$$
total Laplacian   horizontal Laplacian

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2\nabla_H^2 w = 0 \quad \text{internal wave equation.}$$

Notice that if N = 0, no stratification, $\nabla^2\omega = 0$ the motion is irrotational -> surface

gravity waves.

Consider now a plane wave solution in three-dimensions

w = $w_o$cos(kx+ly+mz-ωt)

(k,l,m) wave number

Substituting into the internal wave equation, we obtain the dispersion relationship

$$\omega^2 = \frac{N^2(k^2 + l^2)}{(k^2 + l^2 + m^2)}$$

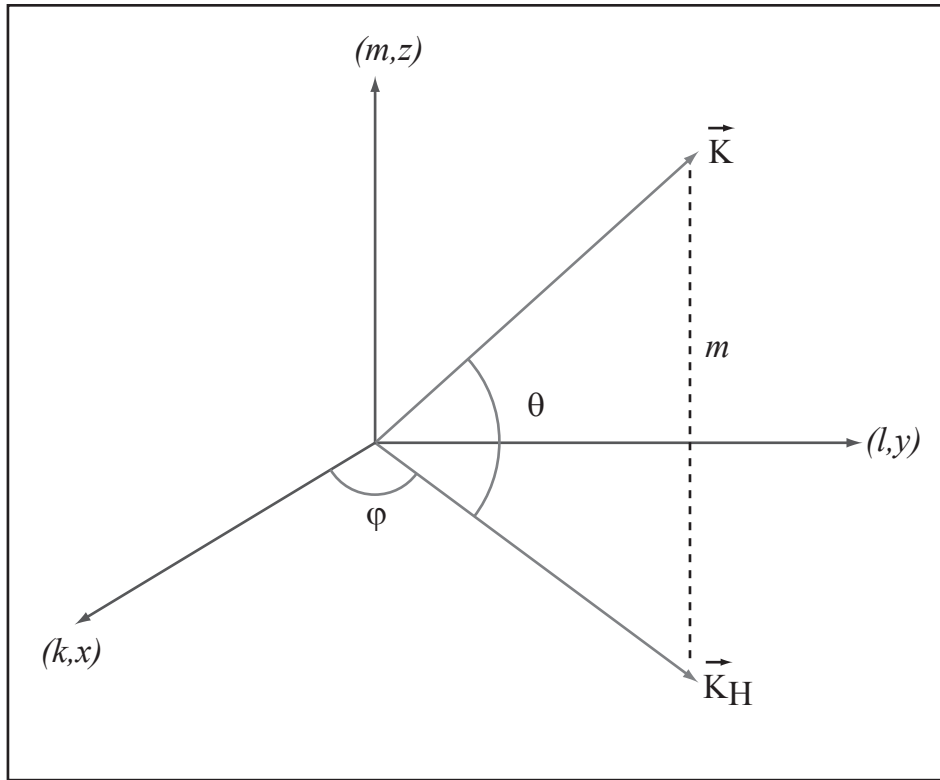$$\text{or} \quad \omega^2 = \frac{N^2 k_H^2}{K^2}$$

$$\omega = \pm N \frac{k_H}{K}$$

Figure 2.

$$\omega = \pm N \cos \theta$$

And the wave numbers are

$$k = K \cos\theta\cos\phi, \quad l = K \cos\theta\sin\phi; \quad m = K \sin\theta$$

The dispersion relation for internal waves is of a quite different character from the dispersion relation for surface waves:

ω of surface waves depends only on the magnitude of K and not on its direction

ω of internal waves is independent of the magnitude of K and depends only on the orientation of the wave vector, i.e. on the angle θ of the wavenumber K with the horizontal.

Consider the displacement ζ along a line of constant phase by definition perpendicular to the phase line
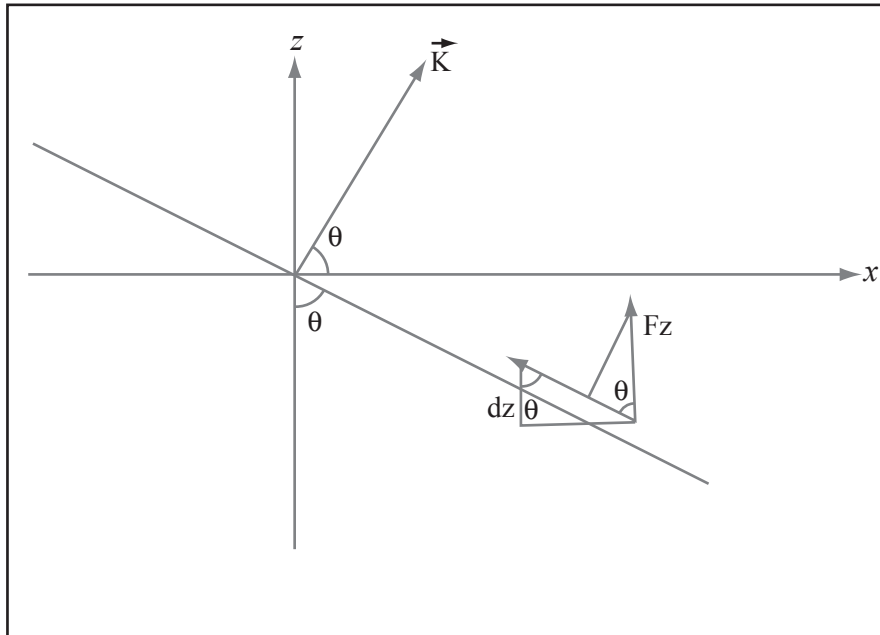
Figure 3.

Thus particle motion is along the wave crests i.e., along the lines of constant phase

$$\frac{d^2\zeta}{dt^2} + N^2\zeta = 0$$

We can see this also considering

$$(u, v, w) = (u_o, v_o, w_o)e^{i(kx+ly+mz-\omega t)}$$

Continuity equation gives

$$ku_o + lv_o + mw_o = 0 \qquad\qquad => \vec{K}\bullet\vec{u} = 0$$

Fluid velocity is perpendicular to the wave vector, i.e. is along the crests of the waves

=> wave motion is transverse
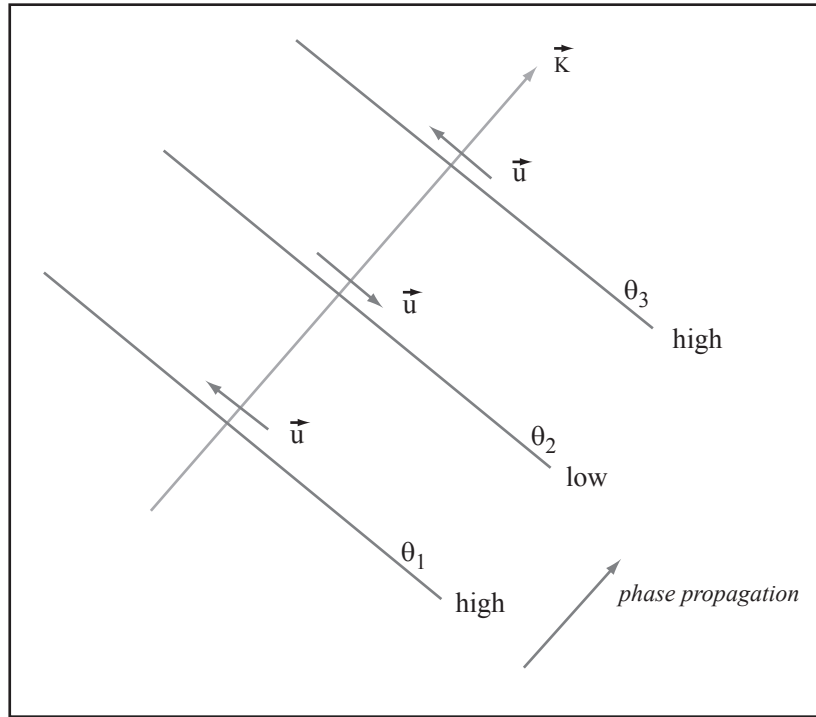
Figure 4.

When the wave vector is horizontal, $m = o$, the particle motion is purely vertical and

$\omega_{max} = N$

A displacement ζ along the phase line gives a vertical displacement

$$dz = \zeta\cos\theta$$

The buoyancy force in the vertical corresponding to the displacement dz is

$$F_z = -N^2 dz = -N^2 \zeta\cos\theta$$

The component of this force along the phase line is

$$F_\zeta = -N^2 \zeta\cos^2\theta$$

and the motion of the particle along the phase line is governed by:

$$\frac{d^2\zeta}{dt^2} = F_\zeta \qquad \rightarrow \qquad \frac{d^2\zeta}{dt^2} + (N^2 \cos^2\theta)\zeta = 0$$

Thus $\pm N\cos\theta$ is the frequency of oscillation of a particle along the phase line. If $\theta = 0$, we have vertical oscillations with frequency N.

Dispersion effects

For internal waves, the surfaces of constant frequency in wavenumber space are the cones θ = constant, i.e.

$$\omega = \text{constant}$$

The phase velocity $|c| = \dfrac{\omega}{K} = \dfrac{N}{K}\cos\theta$ is directed along the wave vector and therefore lies on the cone. The group velocity $c_g = i\dfrac{\partial\omega}{\partial k} + \hat{j}\dfrac{\partial\omega}{\partial l} + \hat{k}\dfrac{\partial\omega}{\partial m}$ is the gradient of ω in wavenumber space and therefore by definition is perpendicular to the surfaces ω = constant. It follows that $\vec{c}_g$ is at right angles with $(K,\vec{c})$.

When the group velocity has an upward component the phase speed has a downward component.

Let us explicitly evaluate the group velocity

Write $\quad \omega^2 = N^2 \dfrac{k^2 + l^2}{k^2 + l^2 + m^2} = \dfrac{N^2 k_H^2}{K^2}$

$$\frac{\partial\omega}{\partial k} = \frac{N}{K}\frac{m^2}{K^2}\frac{k}{k_H} = \frac{N}{K}\sin\theta(\sin\theta\cos\phi)$$

$$\frac{\partial\omega}{\partial k} = \frac{N}{K}\frac{m^2}{K^2}\frac{l}{k_H} = \frac{N}{K}\sin\theta(\sin\theta\sin\phi)$$

$$\frac{\partial\omega}{\partial m} = -N\frac{k_H m}{K^3} = -\frac{N}{K}\sin\theta(\cos\theta)$$

Note

$$\frac{\omega}{m}\frac{\partial\omega}{\partial m} = -\frac{N^2}{K^2}\cos^2\theta < 0$$

The vertical phase velocity is always opposite to the vertical group velocity. Waves propagating their phase upwards will be propagating their energy downwards and viceversa,

or $\quad c_g = \dfrac{N}{K}\sin\theta(\sin\theta\cos\phi,\sin\theta\sin\phi,-\cos\theta)$.

Explanation of the Figure

For internal waves, wave crest AA' has moved to 4 wavelengths downward to the left, the wave group has moved upward parallel to the crests, at right angle to the phase propagation.

For surface gravity waves in deep water, $\vec{c}_g = \dfrac{1}{2}\vec{c}$

wavecrest AA' has again moved 4 wavelengths downward to the left; the wave group has moved in the same direction but at half the speed, that is two wavelengths: AA', with respect to the group, has moved 2 wavelengths.
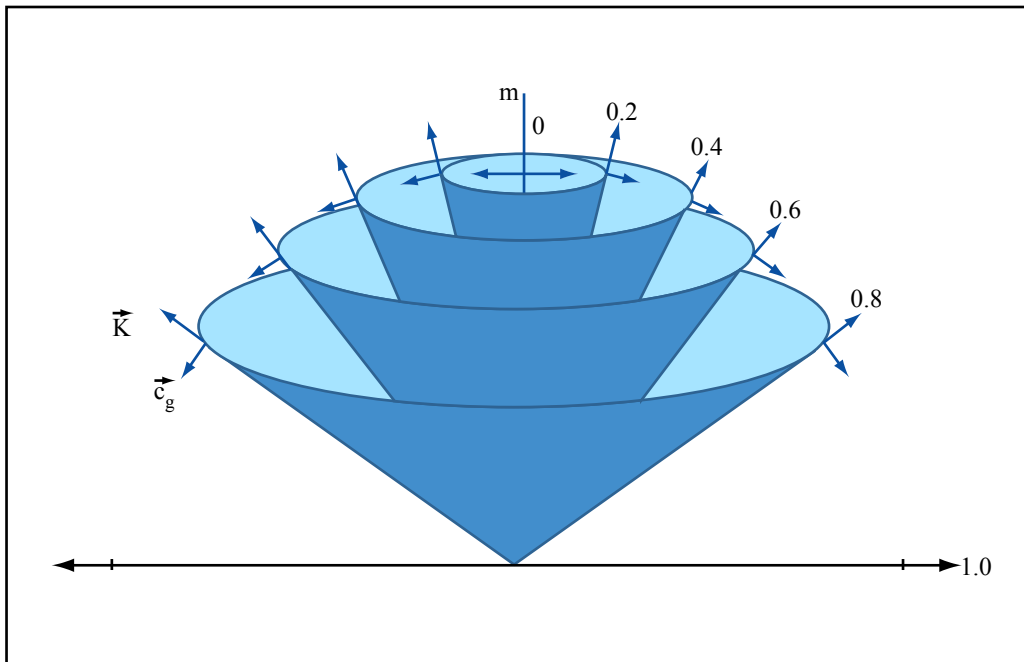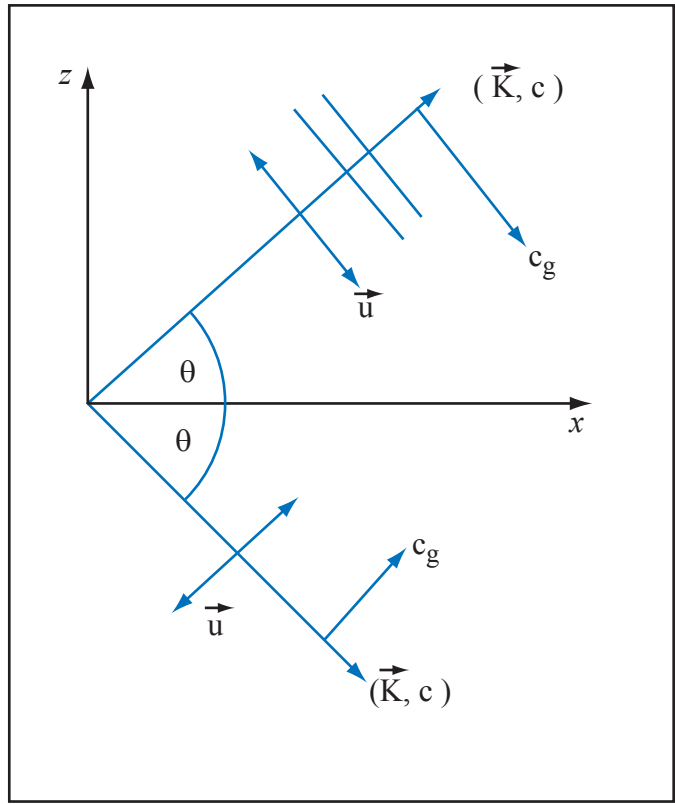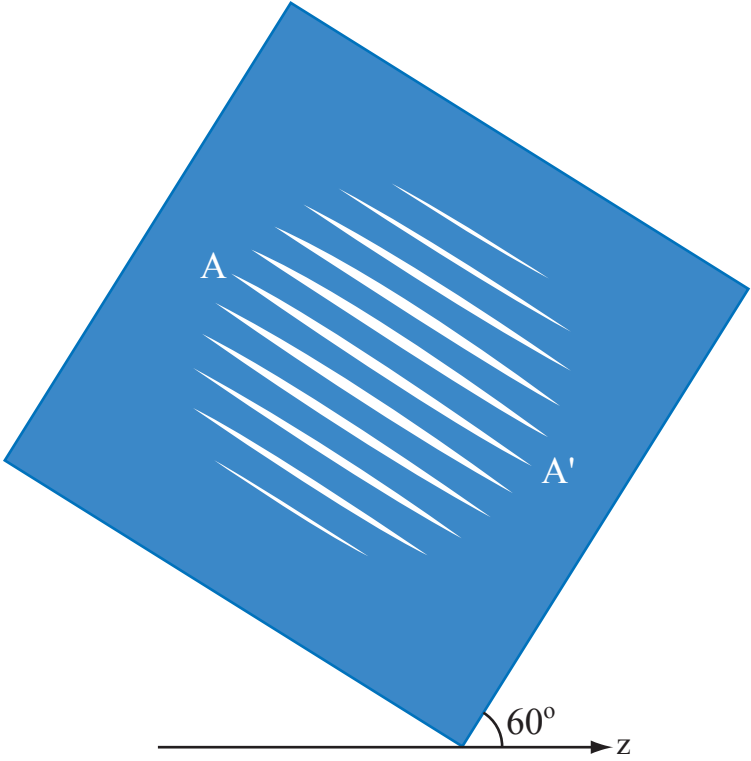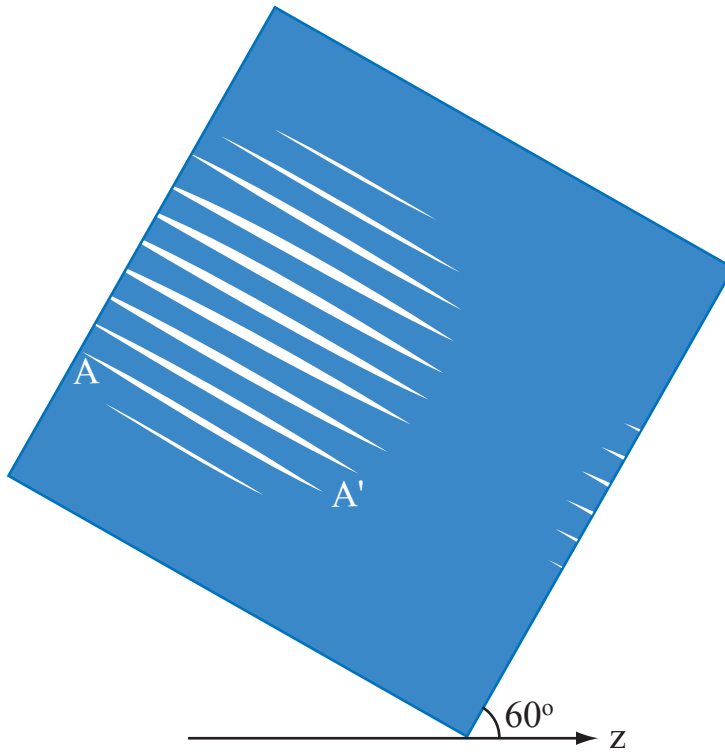
Figure 5.

Figure 6.

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Figure 7a.

Figure 7b.

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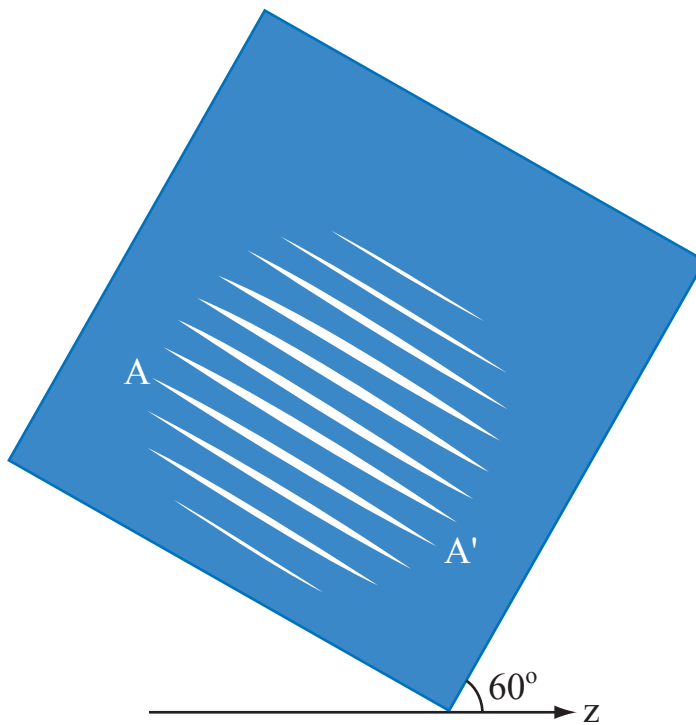
Figure 7c.

12.802 Wave Motion in the Ocean and the Atmosphere
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