Waves are not easy to define, so What is a Wave? Whitham defines a wave as "a recognizable signal that is transferred from one part of a medium to another with recognizable velocity of propagation". A very broad definition, encompassing an enormous range of dynamical systems.

In this course we will consider a number of different types of wave and wave motions occurring in the ocean and the atmosphere, at many different time and space scales. In general, wave-like fluctuations are not exact solutions of the equations of motion, but often represent good approximate solutions of them. Therefore the first step is the appropriate simplification of the equations of motion, which basically involves linearization about some basic (atmosphere, ocean) state at rest or of quasi-steady motion. If the equations are linear, we can superimpose solutions of the equations in order to find solutions to more general initial or boundary conditions. We shall study first such linear waves before relaxing the linearization condition.

<u>Wave Kinematics – definitions</u>

<u>Plane Waves</u>

The simplest form of a wave has periodic variations both in space and time, and the solution to the equations of motion is in the form of a plane wave. This requires that the medium be locally (i.e. on the scale of the wave) homogeneous. Then if ϕ is a field variable (pressure, velocity, etc.)

$$\phi(\vec{x},t) = \mathrm{Re}[Ae^{i(\vec{k}\bullet\vec{x}-\omega t)}] \quad \text{where Re is the real part}$$

$$= |A|\cos(\vec{k}\bullet\vec{x}-\omega t+\tan^{-1}\frac{\mathrm{Im}\,A}{\mathrm{Re}\,A})$$

because As ω is a complex quantity has not only a real amplitude but also a phase factor

So it is much more convenient to work with the complex form and take the real part only at the end. This is possible because we have linearized the equation of motion.

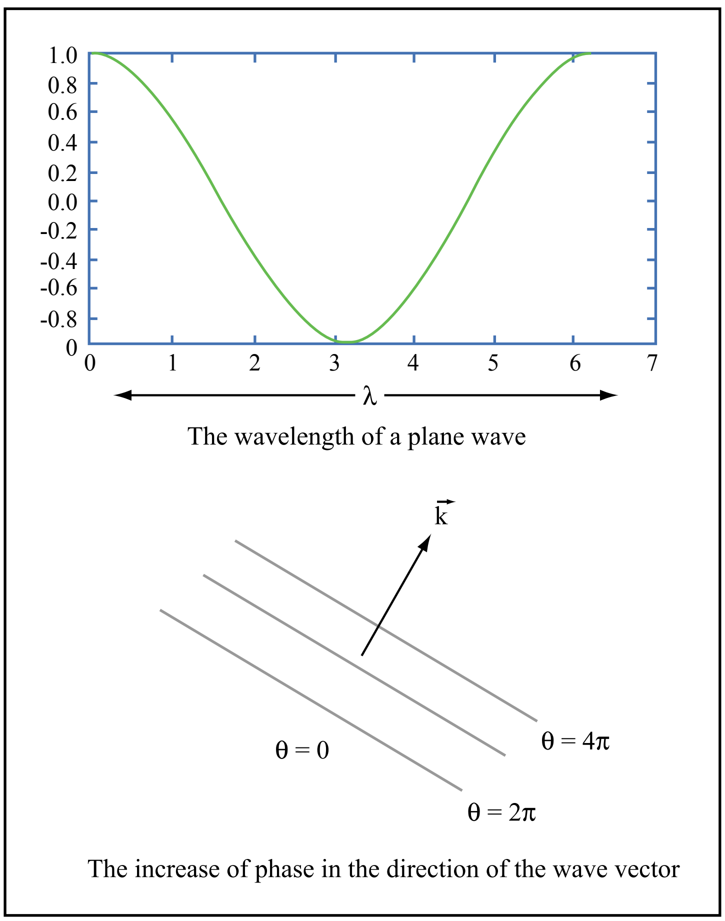
<u>Definitions</u>

- $\vec{x}$ = (x,y,z) (east, south, up)
- $\vec{k}$ = wavenumber (k, l, m)
- phase $\theta = \vec{k} \bullet \vec{x} - \omega t$

I use this convention because if $\omega > 0$ wave "crests and troughs" move in the direction of $\vec{k}$. Surfaces of constant phase:

$$\theta = \vec{k} \bullet \vec{x} - \omega t = kx + ly - \omega t = \text{constant}$$

in two-dimension are planes normal to $\vec{k}$ and moving outward along $\vec{k}$ as t increases.



The wavelength of a plane wave

The increase of phase in the direction of the wave vector

Figure by MIT OpenCourseWare.

Figure 1

If s is the scalar distance along $\vec{k}$

$\vec{k} \bullet \vec{x} = |\vec{K}|\, s$

Is a pure plane wave even achieved in the real world? Looking at swell on a beach it appears to be the zero-order description of the wave field.

The plane wave is a spatially periodic function:

$\phi(|\vec{k}|\, s) = \phi(|\vec{k}|\, [s+\lambda])$ where $|\vec{k}|\, \lambda = 2\pi$ as

$e^{i(|\vec{k}|s)} = e^{i(|\vec{k}|s+2\pi)}$ $e^{i(2\pi)} = 1$

So $\lambda = \dfrac{2\pi}{|\vec{k}|}$ is the wavelength, i.e. the distance along the wave vector between two points with the same phase (Fig. 2).

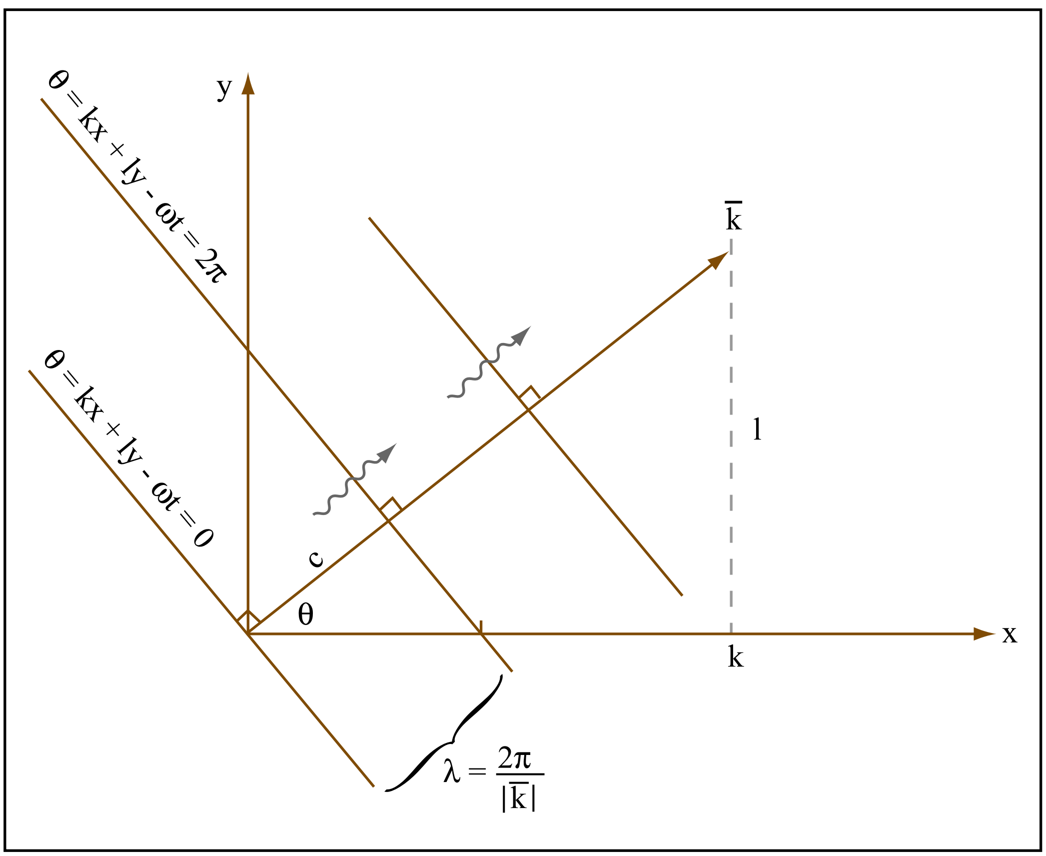


Figure by MIT OpenCourseWare.

Figure 2

At any fixed position the rate of change of the phase with time is $\frac{\partial\theta}{\partial t} = -\omega$

How long do we have to wait until the same phase appears? This shortest time is

$\omega T = 2\pi \rightarrow \quad T = \frac{2\pi}{\omega}$

T is the wave period

As $T = \frac{1}{f}$ the frequency $f = \frac{\omega}{2\pi}$; $\omega = 2\pi f$ the circular frequency.

The phase speed is the speed at which phase planes move along $\vec{k}$, (the speed of propagation of phase along $\vec{k}$)

$$c = \frac{\lambda}{T} = \frac{2\pi}{|\vec{k}|} \bullet \frac{\omega}{2\pi} = \frac{\omega}{|\vec{k}|}$$

which is not a vector. In fact, in two-dimensions the phase speed in x-direction is defined so that, at a fixed y and at constant θ:

$$d\theta = 0 = k\,dx - \omega dt \Rightarrow \quad c_X = \frac{dx}{dt} = \frac{\omega}{k} = -\frac{\partial\theta/\partial t}{\partial\theta/\partial x}$$

So c_x is not $c \cos\theta$ (<u>not</u> a vector) but

$\frac{c}{\cos\theta} = \frac{(\omega/|\vec{k}|)}{(k/|\vec{k}|)} = \frac{\omega}{k}$

$c_x = \frac{\omega}{k}$

$c_y = \frac{\omega}{l}$

$c_z = \frac{\omega}{m}$

Fundamental kinematic equations

$k = \underline{\nabla}\theta \quad \rightarrow$ spatial increase of phase

$\omega = -\dfrac{\partial\theta}{\partial t} \quad \rightarrow$ temporal decrease of phase

The wave $Ae^{i(\vec{k}\bullet\vec{x}-\omega t)}$ is a traveling plane wave. The superposition of two oppositely travelling plane (ω>0) waves:

$$Ae^{i(\vec{k}\bullet\vec{x}-\omega t)} + Ae^{i(-\vec{k}\bullet\vec{x}-\omega t)} = 2Ae^{-i\omega t}\cos(\vec{k}\bullet\vec{x})$$

is a standing wave because crests and troughs do not propagate with time

~o~o~o~

The above is kinematics. In all physical problems, the dynamics impose a functional relation between the wave vector and the frequency. This is the dispersion relation which can be written as

$$\omega = \Omega(\vec{k})$$

which is obtained by requiring the plane wave to be the solution of the linearized, dissipationless equations of motion. The general solution is a superposition of the individual plane waves.

For instance $\phi_t + c\phi_x = 0 \;\; e^{i(kx-\omega t)} \;\; \Rightarrow \omega = \Omega = ck$

The dispersion relation shows how each individual wave in the superposition moves with respect to the other ones, i.e. how from an initial localized package they disperse. Waves are non-dispersive or dispersionless when all waves move with the same phase speed. The package does not change. If the phase speed depends upon the wavenumber and direction then the waves travel each with a different phase and are dispersive, i.e. they do not remain together while propagating through the medium.

Linear superposition of plane waves

In a homogenous medium the general solution is a Fourier integral, which amounts to summing an infinite number of plane waves. If the dispersion relation has n-branches (most general case)

$$\omega = \Omega_j(\vec{k}) \quad j = 1, \ldots, n$$

Then n-initial conditions are required to solve an initial value problem.

This in general defines a "wave bracket"

$$\phi(\vec{x},t) = \sum_{j=1}^{n} \iiint_{-\infty}^{+\infty} A_j(\vec{k}) e^{i[\vec{k}\bullet\vec{x} - \Omega_j(\vec{k})t]} d\vec{k}$$

where the $A_j(\vec{k}))$ are determined by the initial condition. If n = 1, we are in one dimension

$$\omega = \Omega(k)$$

$$\phi(x,t) = \int_{-\infty}^{+\infty} A(k) e^{i[kx - \Omega(k)t]} dk$$

If $\phi(x,o)$, the initial condition, is given:

$$\phi(x,o) = \int_{-\infty}^{+\infty} A(k) e^{ikx} dk \Rightarrow A(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(x,o) e^{-ikx} dx$$

If furthermore $\Omega = ck$ then

$$\phi(x,t) = \int_{-\infty}^{+\infty} A(k) e^{i(kx - ckt)} dk = \int A(k) e^{ik(x-ct)} dk = \phi(x - ct, o)$$

The initial condition $\phi(x,o)$ translates towards $x>o$ at speed c without changing shape. For homogeneous media, to solve for the wave packet:

1) Find the dispersion relation; 2) deduce the $A_j(\vec{k})$ from the initial condition; 3) evaluate the Fourier integrals; 3) is most often difficult to evaluate.

Useful approximate method: In general the envelope modulating the individual plane waves is a wave packet comprising many individual plane waves with nearby wave-numbers and phase speeds. The "group velocity" is therefore the velocity of the "group" i.e. of the wave packet and we shall see it is the velocity with which energy (not phase) propagates.

$$\phi(x,o) = a(x)e^{ik_o x}$$

As a preview, let us consider a one-dimensional example with the special initial condition $\phi(x,0) = a(x)e^{ik_o x}$

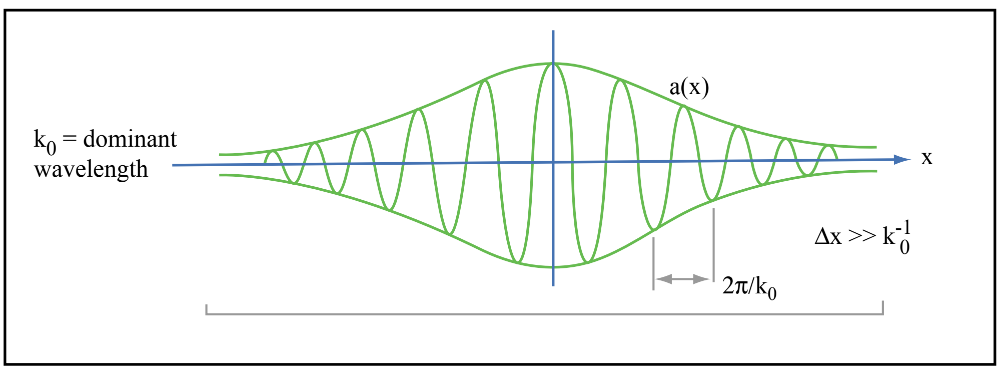


Figure by MIT OpenCourseWare.

Figure 3

This represents a slowly modulated plane wave with envelope a(x). We can always write

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk; \quad A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(x,0)e^{-ikx}dx$$

and so

$$A(k) = \int_{\infty}^{\infty} a(x)e^{i(k_o-k)x}dx; \quad a(x) = \int_{-\infty}^{\infty} A(k)e^{-i(k-k_o)x}dk$$

In this last integral, the contribution to the integral itself is mostly from the regions where the quantity $(k_0\text{-}k)x$ is small. In fact, where this quantity is large, $e^{i(k_0-k)x}$ oscillates rapidly and the integrated parts cancel each other. Moreover, $a(x) = 0$ for $x >> \Delta x$. So, $A(k)$ is centered around k_0 and peaked there for this special choice of $\phi(x,o)$.

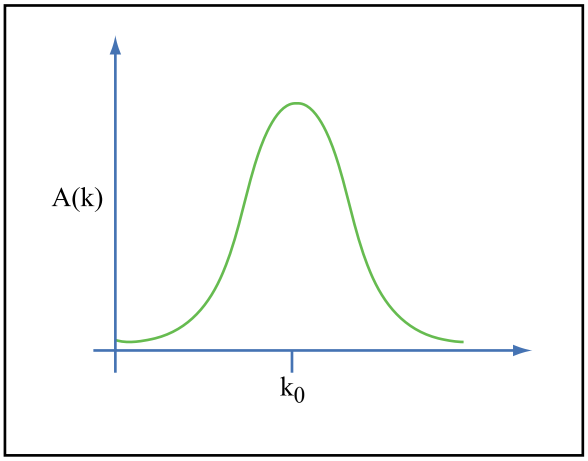


Figure by MIT OpenCourseWare.

Figure 4

The modulated plane wave is said to be a 'narrow band signal'.

We can evaluate $\phi(x,t)$ by expanding $\Omega(k)$ in a Taylor series about k_o:

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\Omega(k)t)}dk$$

$$\simeq \int_{-\infty}^{\infty} A(k)e^{i(kx-\Omega(k_o)t-(k-k_o)\left.\frac{\partial\Omega}{\partial k}\right|_{k=k_o}t)}dk$$

$$\simeq \int_{-\infty}^{\infty} A(k)e^{i(kx-\Omega(k_o)t-(k-k_o)\left.\frac{\partial\Omega}{\partial k}\right|_{k=k_o}t)}e^{ik_ox-ik_ox}dk$$

$$= e^{i[k_o x - \Omega(k_o)t]} \int_{-\infty}^{\infty} A(k) e^{i(k-k_o)(x - \frac{\partial \Omega}{\partial k}\big|_{k=k_o} t)} dk$$

as $\qquad a(x) = \int_{-\infty}^{\infty} A(k) e^{i(k-k_o)} dk$

That is

$$\phi(x,t) = e^{i[k_o x - \Omega(k_o)t]} a(x - \frac{\partial \Omega}{\partial k}\big|_{k=k_o} t)$$

The modulating envelope moves at a velocity $\partial\Omega/\partial k|_{k=k_o}$, defined by the dispersion relation $\omega = \Omega(k)$. This velocity is called the *group velocity*

$$c_g = \frac{\partial \Omega}{\partial k}\big|_{k=k_o}$$

and is not, in general, equal to the phase speed $c = \omega/k$ of the modulated plane wave. Therefore, the dominant wavelength $\lambda=2\pi/k_o$ has two speeds associated with it. They are the phase speed $c=\omega/k_o$ and the group velocity $c_g = \partial\Omega/\partial k|_{k=k_o}$. The modulating envelope thus moves through the underlying plane wave rather than with it.

Method of Stationary Phase

The restriction to narrow band processes is illustrative but not necessary. Consider more generally

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i[kx - \Omega(k)t]} dk$$

Define

$$\theta(k;x,t) \equiv kx/t - \Omega(k) \quad \text{the phase}$$

Then

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{it\theta(k;z,t)} dk$$

Suppose the wave packet has a complicated shape; that is at long distance and time from the place of generation, the waves originally "packed" have dispersed and the rapid oscillations of $e^{i\theta t}$ as $t \to +\infty$ cancel each other like in figure 5. This means that

$$\lim_{t \to \infty} \int_{-\infty}^{+\infty} A(k) e^{i\theta t} dk = 0$$

Then there is very little contribution to $\phi(x,t)$ unless there is a point where $\theta(k,x,t)$ has no variation with k, that is there is a k_o where the phase is stationary and we can apply the method of stationary phase, i.e. expansion around k_o:

$$\phi(x,t) \simeq \int_{-\infty}^{+\infty} A(k_o) e^{it\left[\theta(k_o)+\theta'(k_o)(k-k_o)+\theta''(k_o)\frac{(k_o-k_o)^2}{2}\right]}$$

Then at a given (x,t) the greatest contribution to $\phi(x,t)$ is from the wave number k_o where $\theta'(k_o) \equiv 0$.

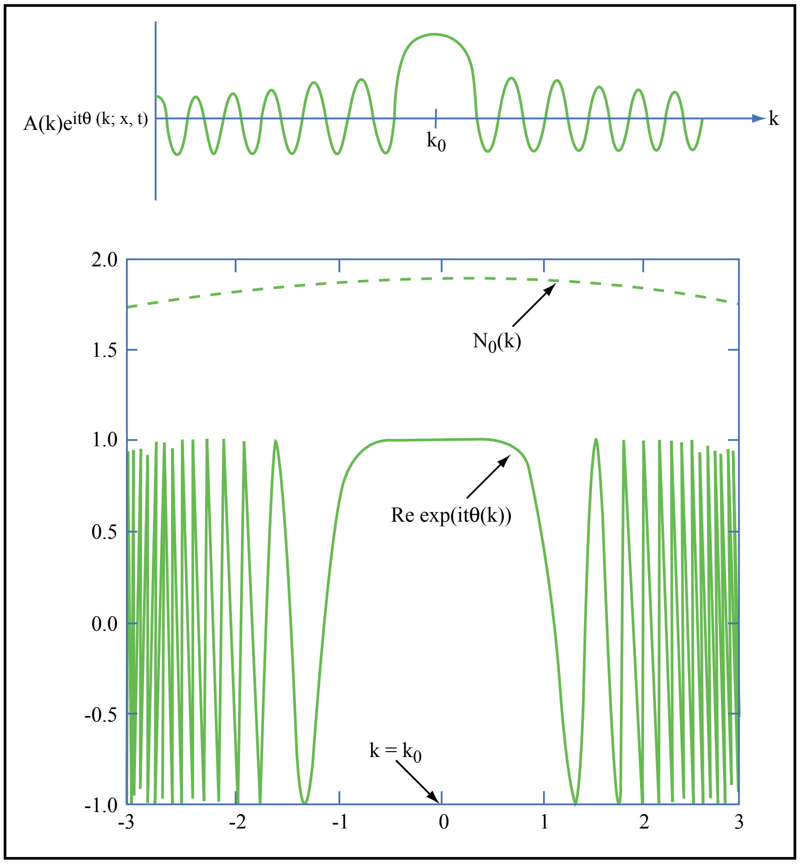


Figure by MIT OpenCourseWare.

Figure 5

As $\theta(k,x,t) = k\frac{x}{t} - \Omega(k)$ we have $\frac{\partial\theta}{\partial k}\Big|_{k=k_o} = \frac{x}{t} - \frac{\partial\Omega}{\partial k}\Big|_{k=k_o} = 0$ which means that the wavenumber k_o making the biggest contribution to $\phi(k,t)$ is that for which $\frac{\partial\Omega}{\partial k}\Big|_{k=k_o} = \frac{x}{t} = C_g$

The solution is then

$$\phi(x,t) = A(k_o)e^{it\theta(k_o)}\int_{-\infty}^{+\infty} e^{i(k-k_o)^2\frac{\theta''(k_o)t}{2}}\,dk$$

$$\frac{\partial\theta}{\partial k}\Big|_{k=k_o} = 0$$

As $\int_{-\infty}^{+\infty} e^{-\alpha z^2} = (\frac{\pi}{\alpha})^{1/2}$ with $z^2 = (k - k_o)^2$

$$\alpha = \frac{\theta''(k_o)t}{2i}$$

then $\phi(x,t) \simeq A(k_o) e^{it\theta(k_o)} [\frac{2\pi i}{t\theta''(k_o)}]^{1/2}$

or $\phi(x,t) \simeq A(k_o) e^{i[k_o x - \Omega(k_o)t]} [\frac{2\pi i}{t\theta''(k_o)}]^{1/2}$

The solution is a slowly-modulated plane wave whose wavenumber k_o has $c_g = x/t$.

The solution is valid only for large t as it requires the rapid oscillation of $e^{i\theta(k)t}$ to cancel out except those where $c_g = \frac{\partial \Omega}{\partial k} \equiv \frac{x}{t}$.

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