Lecture 11

11.1 Administration

- Collect problem set.
- Distribute problem set due October 23, 2004.

11.2 2nd law of thermodynamics

(Jim – You have a note for yourself here that reads "Fill in full derivation, this is a sloppy treatment").

For completeness, I feel obligated to go through the derivation of the 2nd law of thermodynamics as well. Kundu and Cohen actually do a god job with this one, so all I'm going to do is mimic their derivation. The change in entropy is given by

$$T\,dS = de + p\,d\alpha, \quad \alpha = \frac{1}{\rho} \tag{11.1}$$

$$\Rightarrow \quad T\frac{DS}{Dt} = \frac{De}{Dt} + p\frac{D\alpha}{Dt} \tag{11.2}$$

Remember, the laws of thermodynamics are empirical. This definition of entropy (equation 11.1) "just is." Also recall that de is the internal component, while $p\,d\alpha$ is the work component, and the differential form (equation 11.2) is achieved by dividing by dt and taking the limit. Next, recall that

$$\frac{1}{x^2}\frac{dx}{dt} = -\frac{d\frac{1}{x}}{dt} \Rightarrow \frac{D\frac{1}{\rho}}{Dt} = -\frac{1}{\rho^2}\frac{D\rho}{Dt} \tag{11.3}$$

Hence

$$T\frac{DS}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} \tag{11.4}$$

We already know that

$$\frac{De}{Dt} = -\frac{1}{\rho}\nabla\cdot\mathbf{q} - \frac{p}{\rho}\nabla\cdot\mathbf{u} + \frac{\phi}{\rho} \qquad \text{(energy equation)} \tag{11.5}$$

and

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u} \qquad \text{(continuity equation)} \tag{11.6}$$

Thus, substitution yields

$$T\frac{DS}{Dt} = -\frac{1}{\rho}\nabla\cdot\mathbf{q} - \frac{p}{\rho}\nabla\cdot\mathbf{u} + \frac{\phi}{\rho} + \frac{p}{\rho}\nabla\cdot\mathbf{u} \tag{11.7}$$

$$T\frac{DS}{Dt} = -\frac{1}{\rho}\nabla\cdot\mathbf{q} + \frac{\phi}{\rho} \tag{11.8}$$

We can expand this a bit more by noting that

$$\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) = \mathbf{q} \cdot \nabla \left(\frac{1}{T}\right) + \frac{1}{T}\nabla \cdot \mathbf{q} \tag{11.9}$$

And again using $\frac{d(1/x)}{dy} = -\frac{1}{x^2}\frac{dx}{dy}$, this becomes

$$\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) = -\frac{\mathbf{q}}{T^2} \cdot \nabla T + \frac{1}{T}\nabla \cdot \mathbf{q} \tag{11.10}$$

$$\Rightarrow \quad -\frac{1}{T}\nabla \cdot \mathbf{q} = -\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) - \frac{\mathbf{q}}{T^2} \cdot \nabla T \tag{11.11}$$

Substituting this results into 11.8 yields

$$\rho\frac{DS}{Dt} = -\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) - \frac{\mathbf{q}}{T^2} \cdot \nabla T + \frac{\phi}{T} \tag{11.12}$$

Now recall Fourier's Law

$$\mathbf{q} = -K\nabla T \tag{11.13}$$

Substitution then yields

$$\rho\frac{DS}{Dt} = -\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{K}{T^2}\nabla^2 T + \frac{\phi(\mu)}{T} \tag{11.14}$$

Here the first term on the RHS is the entropy gain due to reversible heat transfer. The second term on the RHS is the entropy gain due to nonreversible heat conduction. The final term is the entropy gain due to viscous generation of heat. Because the 2nd law tells us that entropy production from irreversible processes must be positive, then we require $K > 0$ (thermal diffusivity), and $\mu > 0$ (viscosity).

11.3 Summary of equations (non-Boussinesq)

Continuity

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0 \tag{11.15}$$

Momentum

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{u} + \frac{\mu}{3}\nabla(\nabla \cdot \mathbf{u}) \tag{11.16}$$

Heat (assuming a linear equation of state)

$$\rho C_v\frac{DT}{Dt} = -p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q} \tag{11.17}$$

State

$$p = \rho RT, \qquad \text{or} \qquad \rho = \rho_o\left(1 - \alpha_T(T - T_o) + \alpha_S(S - S_o)\right) \tag{11.18}$$

The first equation is the ideal gas law, as commonly applied to the atmosphere. The second equation is the EOS for the ocean, where α_T and α_S are empirical. This also requires a salt equation, typically something like $\frac{DS}{Dt} = \gamma\nabla^2 S$.

Entropy (Jim – K is not used here, you crossed out the ones where it was)

$$\mu > 0, K > 0 \tag{11.19}$$

Hence, altogether, there are 6 (7 for ocean) equations and 6 (7) unknowns.

11.4 Summary of equations (Boussinesq)

$$\rho = \rho_o + \rho'(x, y, z, t) \tag{11.20}$$

Continuity (to first order)

$$\nabla \cdot \mathbf{u} = 0 \tag{11.21}$$

Momentum (traditional to leave primes off)

$$\rho_o \frac{D\mathbf{u}}{Dt} = \rho' \mathbf{g} - \nabla p' + \mu \nabla^2 \mathbf{u} \tag{11.22}$$

Heat (crap for atmosphere, not so great for ocean)

$$\frac{DT}{Dt} = K \nabla^2 T, \qquad K = \frac{k}{\rho C_p} \tag{11.23}$$

State

$$p = \rho R T, \qquad \text{or} \qquad \rho = \rho_o(1 - \alpha_T(T - T_o) + \alpha_S(S - S_o)) \tag{11.24}$$

Again, the first equation is the ideal gas law, and the second equation is a simple EOS for the ocean. A salt equation, typically something like $\frac{DS}{Dt} = \gamma \nabla^2 S$, is required.

Entropy

$$\mu > 0, K > 0 \tag{11.25}$$

11.5 Summary of equations (Euler)

(Jim – You have a note to yourself to clean this section up)

$$\mu = 0, \qquad \nabla \cdot \mathbf{u} = 0 \text{ (usually, but not always)} \tag{11.26}$$

Continuity (typically, can be full continuity equation)

$$\nabla \cdot \mathbf{u} = 0 \tag{11.27}$$

Momentum (traditional to leave primes off)

$$\rho_o \frac{D\mathbf{u}}{Dt} = \rho' \mathbf{g} - \nabla p' \tag{11.28}$$

State (typically, can be any)

$$\rho = \rho_o \tag{11.29}$$

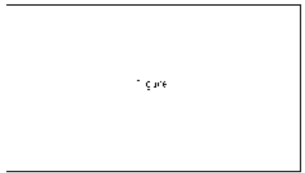

Figure 11.1: (`fig:Lec11Circulation1`) The circulation is the sum of the component of velocity tangent to C summing all around C.

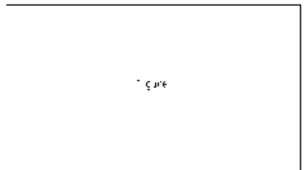

Figure 11.2: (`fig:Lec11Circulation2`) For steady, uniform flow, $\Gamma = 0$.

11.6 Vorticity – At last!

Vortex motion is motion in circular streamlines. But, just because you have closed streamlines does not necessarily mean you will have fluid blobs rotating about their centers. Odd though it may seem, one can have both rotational and irrotational vortices. An example of a rotational vortex is solid body rotation where we have

$$u_\theta = \Omega_o r, \qquad \Omega_o \equiv \text{angular velocity} \tag{11.30}$$

An example of an irrotational vortex is the point, or line, vortex, where we have

$$u_\theta = \frac{\Gamma}{2\pi r}, \qquad \Gamma \equiv \text{circulation} \tag{11.31}$$

What is this thing called circulation? Mathematically, it is defined as

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{s} \tag{11.32}$$

Γ is the sum of the component of velocity tangent to C summing all around C (see figure 11.1). So, if steady, uniform flow, $\Gamma = 0$ because all $\mathbf{u} \cdot d\mathbf{s}$ will cancel one another out (see figure 11.2).

Stokes, who was so helpful sorting out our 4th order tensor problems, has a theorem that states:

$$\oint_C \mathbf{u} \cdot d\mathbf{s} = \iint_A (\nabla \times \mathbf{u}) \cdot d\mathbf{A} \tag{11.33}$$

This related the line integral (about a closed curve) of a vector field to a surface integral of the vector field where the surface is a "capping" surface of the closed curve (see figure 11.3). Stokes theorem is a lot like the divergence theorem (they are linked by exterior calculus), and is sufficiently neat to warrant some time in class. Just like the divergence theorem, we'll take our capping surface and closed surface, and break each into polygon chunks (see figure 11.4). Assert that circulation around a closed loop is same as the sum of circulations around the polygons.

$$\Rightarrow \quad \oint_{\tilde{C}} \mathbf{u} \cdot d\mathbf{s} = \sum_{i=1}^{N} \oint_{C_i} \mathbf{u} \cdot d\mathbf{s} \tag{11.34}$$

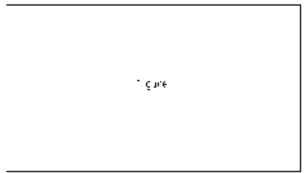

Figure 11.3: (`fig:Lec11Circulation3`) A closed surface defined by C, and its "capping" surface, A.

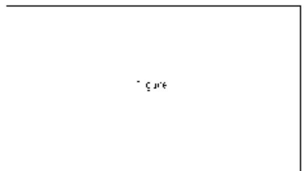

Figure 11.4: (`fig:Lec11Circulation4`) The closed surface defined by C, and its capping surface, A, broken into polygon chunks.

I'll try and convince you of this in the same way as I did for the divergence theorem. See figure 11.5. Consiter the common branch between C_1 and C_2. When calculating the component of the circulation in C_1 that corresponds to the branch between A and B you have a term like

$$\int_A^B \mathbf{u} \cdot d\mathbf{s} \tag{11.35}$$

When calculating the component of the circulation in C_2 that corresponds to the same branch you get

$$\int_B^A \mathbf{u} \cdot d\mathbf{s}, \quad \text{since} \quad \int_A^B \mathbf{u} \cdot d\mathbf{s} = -\int_B^A \mathbf{u} \cdot d\mathbf{s} \tag{11.36}$$

Which means that the contribution to the circulation from the common sides of neighboring polygons will be 0. The fact that there will be canceling contributions on all sides except the sides common with the closed loop confirms that

$$\oint_{\tilde{C}} \mathbf{u} \cdot d\mathbf{s} \quad = \quad \sum_{i=1}^{N} \oint_{C_i} \mathbf{u} \cdot d\mathbf{s} \tag{11.37}$$

We now apply a trick similar to the one we use for the divergence theorem and multiply and divide the RHS by ΔA_i, the area of the ith face.

$$\oint_{\tilde{C}} \mathbf{u} \cdot d\mathbf{s} \quad = \quad \sum_{i=1}^{N} \left[\frac{1}{\Delta A_i} \int_{C_i} \mathbf{u} \cdot d\mathbf{s} \right] \Delta A_i \tag{11.38}$$

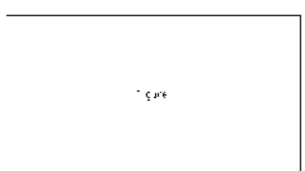

Figure 11.5: (`fig:Lec11Circulation5`) Note $\Gamma_1 = \oint \mathbf{u} \cdot d\mathbf{s_1}$ and $\Gamma_2 = \oint \mathbf{u} \cdot d\mathbf{s_2}$.

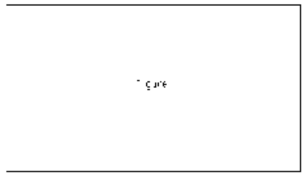

Figure 11.6: (`fig:Lec11SolidBodyRotation`) Schematic of solid body rotation. Note that $u_\theta = \Omega_o r$ and $u_r = 0$.

The quantity in the brackets (for $\Delta A_i \gg 0$) is, by definition, the curl of $\mathbf{u}$. The curl is the circulation per area as area goes to zero, and has the direction of normal to that area.

$$\lim_{N\to\infty,\Delta A_i\to 0} \oint_{\tilde{C}} \mathbf{u}\cdot d\mathbf{s} = \lim_{N\to\infty,\Delta A_i\to 0} \sum_{i=1}^{N} \mathbf{n}_i \cdot (\nabla\times\mathbf{u}) A_i \tag{11.39}$$

$$\oint_{\tilde{C}} \mathbf{u}\cdot d\mathbf{s} = \iint_A (\nabla\times\mathbf{u})\cdot d\mathbf{A} \tag{11.40}$$

(Jim – I'm not quite sure how to stack the limits so they do not string out like they do now). Stokes theorem tells us that the line integral of the tangental part of a vector function around a closed path is equal to the surface integral of the normal component of the curl of the vector function over any capping surface of the closed path. Neat! Divergence is the flux per volume while curls is the circulation per area.

11.7 Solid body rotation

Where are we now? Solid body rotation is where the velocity is proportional to the radius of the streamlines (see figure 11.6). The vorticity in polar coordinates is given by

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \tag{11.41}$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega_o) = \frac{1}{r}(2r\Omega_o) = 2\Omega_o \tag{11.42}$$

This implies that the time it takes a particle to rotate about its own center is the same as it takes the particle to rotate about the center of the SBR.

And how does this compare with the circulation?

$$\Gamma = \oint_C \mathbf{u}\cdot d\mathbf{s} \tag{11.43}$$

Consider a radius, r

$$\Gamma = \int_0^{2\pi} u_\theta r\, d\theta = u_\theta r 2\pi = (2\Omega_o)\pi r^2 = \omega_z \pi r^2 \tag{11.44}$$

This is just the vorticity times the area, as it should be. Check this with Stokes.

$$\Gamma = \iint_A (\nabla_{\theta,r}\times\mathbf{u})\cdot d\mathbf{A} = \iint_A \omega_z dA = \omega_z \pi r^2 \tag{11.45}$$

As it should be. Note from the picture of SBR (figure 11.6) that fluid elements do not deform. $ABCD$ is not deformed relative to $A'B'C'D'$. We know that it is the deformation rate of fluid

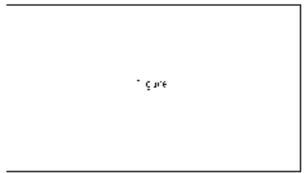

Figure 11.7: (`fig:Lec11PotentialVortex1`) Schematic of an irrotational (potential) vortex. Here $u_\theta = \frac{c}{r}$ and $u_r = 0$.

blobs that leads to viscosity, so since we have no deformation we can conclude that SPR is an inviscid phenomenon – Although viscosity will be important during spin up and at the edge of the SBR.

In SBR, the u_θ's at large r are much greater than the u_θ's at small r. At first glance one might think we should then have low pressure at large r and high pressure at small r. From Bernoulli:

$$\rho\frac{1}{2}u_{\theta_{\text{fast}}}^2 + p_1 \quad = \quad \rho\frac{1}{2}u_{\theta_{\text{slow}}}^2 + p_2 \tag{11.46}$$

But remember, $\rho\frac{1}{2}u_\theta^2 + p + gz$ is only constant everywhere for inviscid, irrotational flow. SBR is not irrotational, so

$$\begin{aligned} \rho\frac{1}{2}u_\theta^2 + p &= \text{constant on a streamline} && (11.47)\\ &\Rightarrow \text{constant} = f(r) && (11.48) \end{aligned}$$

11.8 Irrotational (potential) vortex

An irrotational (potential) vortex is where the velocity falls off as $\frac{1}{r}$ (see figure 11.7). Vorticity here is

$$\begin{aligned} \omega_z &= \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \\ &= \frac{1}{r}\frac{\partial}{\partial r}c \\ &= \frac{0}{r}\,! \end{aligned} \tag{11.49}$$

We have closed, circular streamlines and no vorticity!?! Actually we have zero vorticity everywhere except at the origin. There $\omega_z = \frac{0}{0}$, so we're not really sure what is going on.

We can calculate the circulation of a loop enclosing the center of the vortex at some radius r:

$$\Gamma \quad = \quad \int_c \mathbf{u} \cdot d\mathbf{s} \tag{11.50}$$

$$\Gamma \quad = \quad \int_0^{2\pi} u_\theta r\, d\theta = u_\theta r 2\pi = \frac{c}{r} r 2\pi = 2\pi c \tag{11.51}$$

This tells us that Γ is constant – not a function of r as in the case of SBR. It also says that $c = \frac{\Gamma}{2\pi} \Rightarrow u_\theta = \frac{\Gamma}{2\pi r}$. Consider Stokes' theorem for a closed curve that includes the origin:

$$\Gamma = \int_c \mathbf{u} \cdot d\mathbf{s} = \iint_A \omega \cdot dA = 2\pi c \tag{11.52}$$

For the area integral to be nonzero, ω_z must be nonzero somewhere within the closed curve. Since we know $\omega_z = 0$ everywhere except the origin, ω_z must be nonzero only there. Then what should

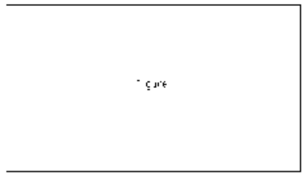

Figure 11.8: (`fig:Lec11PotentialVortex2`) The circulation of any closed curve that does not contain the origin is zero. This is a schematic of such a closed curve.

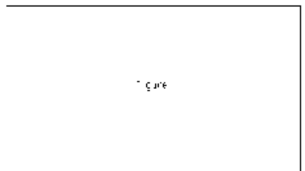

Figure 11.9: (`fig:Lec11PotentialVortex3`) Deformation of a fluid blob in a potential vortex.

the magnitude be? At the origin, dA is infinitely small, so ω_z must be infinitely large. A potential vortex is irrotational everywhere except as the origin, where the vorticity is infinite. A delta function of vorticity.

The circulation of any closed curve that contains the vortex is the same ($\Gamma = 2\pi c$), but the circulation of and closed curve that does not contain the origin is zero (see figure 11.8 for diagram):

$$\begin{aligned}
\Gamma &= \int_A^C \mathbf{u} \cdot d\mathbf{s} + \int_C^D \mathbf{u} \cdot d\mathbf{s} + \int_D^B \mathbf{u} \cdot d\mathbf{s} + \int_B^A \mathbf{u} \cdot d\mathbf{s} && (11.53)\\
&= \int_C^D \mathbf{u} \cdot d\mathbf{s} + \int_B^A \mathbf{u} \cdot d\mathbf{s} && (11.54)\\
&= \frac{c}{r_2} r_2 \Delta\theta - \frac{c}{r_1} r_1 \Delta\theta && (11.55)\\
&= c\Delta\theta - c\Delta\theta && (11.56)\\
&= 0 && (11.57)
\end{aligned}$$

Note that in the first step, the two integral terms drop out because $u_r = 0$.

Consider next the deformation of a fluid blob in a potential vortex as is seen in figure 11.8. There is a deformation of the blob, implying that shear stresses exist. A shear stress implies viscous stresses and viscous stresses imply vorticity. Yet we have no vorticity here. What gives? The rotation of the blob about its own center is exactly balanced by the rotation of the blob about the center of the vortex. Stated in a different manner, the rotation of one axis is exactly balanced by the rotation of the other axis. This, combined with the results from solid body rotation, lead to two important points.

- Irrotational flow need not mean there are no viscous stresses (potential vortex).
- Absence of viscous stresses need not mean absence of vorticity (solid body rotation).

Show film.

11.9 Real vortices

I want to briefly say something about combining SBR vortices and potential vortices. "Real vortices" (tornadoes, hurricanes, etc.) behave like SBR near the core, and like a potential vortex

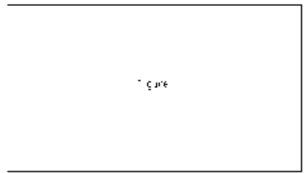

Figure 11.10: (`fig:Lec11RealVortex`) ω_z and u_z profiles for "real vortices".

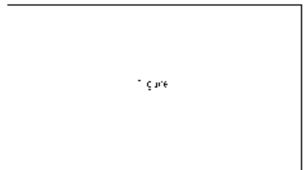

Figure 11.11: (`fig:Lec11RankineVortex`) ω_z and u_z profiles for a Rankine Vortex.

far from the core (see figure 11.10). We can approximate this behavior by defining a vortex with SBR out to some distance R, and potential flow beyond that (see figure 11.11). This is called a Rankine vortex. In read vortices, what sets R?

11.10 Reading for class 12

KC01: 5.4 - 5.5, meaning of $(\omega \cdot \nabla)\mathbf{u}$ section in 5.6

(Jim - I couldn't tell if this last bit was from KC or elsewhere).