Lecture 11: Introduction to Constitutive Equations and Elasticity

Previous lectures developed the concepts and mathematics of stress and strain. The equations developed in these lectures such as Cauchy's formula and Newton's second law were derived from laws of physics and are valid for all continuous media. The next couple of lectures explain the relationships between stress and strain. These relationships are called constitutive equations and apply to only specific idealized materials. For example, constitutive equations characterize the behavior of viscous fluids and perfectly elastic solids.

1. Elasticity

Introduction

Elasticity is an important phenomenon in Earth Science because rocks behave elastically at conditions prevalent in most of the upper lithosphere: low temperature and pressure. Elastic behavior means that a rock deforms instantly under the application of a load and that the strain is linearly proportional to stress. Once the load is removed, all of the strain is recovered. These characteristics of elastic behavior are shown graphically below.
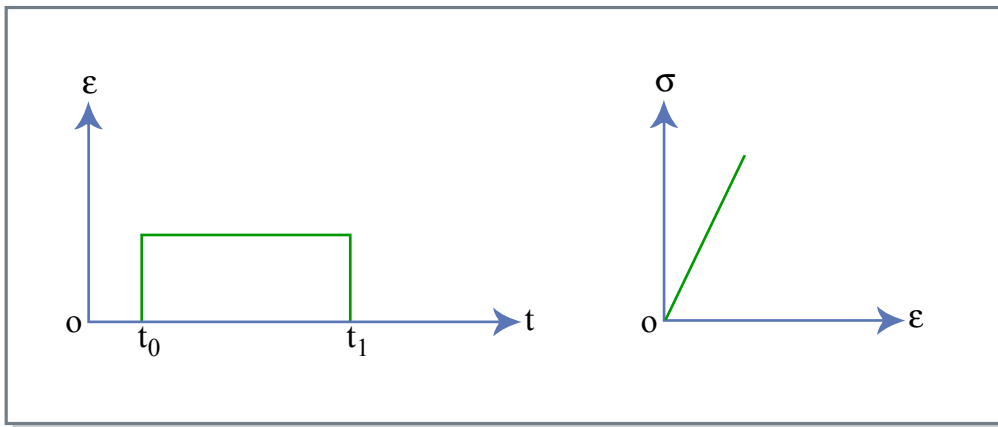

Figure 11.1

Figure by MIT OCW.

The picture on the left shows strain as a function of time for an elastic body under a constant stress applied at t_0. The picture on the right shows that strain is linearly proportional to stress in an elastic body.

<u>Mathematics</u>

Previous lectures explained that second-order tensors provide a relationship between two first-order tensors. Lecture V, for example, explained that stress is a second-order tensor that relates the normal vector of a plane to the traction vector on that plane. Constitutive equations relate two second-order tensors by using a fourth-order tensor. The general form of the equation for elasticity, for example, is given by

$$\tau_{ij} = c_{ijkl}\varepsilon_{kl}$$

where c_{ijkl} is the fourth-order elastic modulus tensor. It has 81 components.

There are too many components of the elastic modulus tensor to work with easily. However, it can be reduced to fewer components using arguments of symmetry and energy. The fact that the stress and strain tensors are symmetric reduces the number of components to 36. This number can be further reduced to 21 by considering constraints on strain energy. For a full treatment of the strain energy argument, see pages 56-58 in <u>Rheology of the Earth</u> by Ranalli. Lastly, by assuming the material is isotropic, the elastic modulus tensor can be reduced to

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where λ and μ are called Lamé parameters. Using this form of the elastic modulus tensor, the constitutive equation for elasticity is written:

$$\tau_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

Equations for the normal stresses are written:

$$\tau_1 = (\lambda + 2\mu)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3$$
$$\tau_2 = \lambda\varepsilon_1 + (\lambda + 2\mu)\varepsilon_2 + \lambda\varepsilon_3$$
$$\tau_3 = \lambda\varepsilon_1 + \lambda\varepsilon_2 + (\lambda + 2\mu)\varepsilon_3$$

The equations for the shear stresses are written:

$$\tau_{12} = 2\mu\varepsilon_{12}$$
$$\tau_{13} = 2\mu\varepsilon_{13}$$
$$\tau_{23} = 2\mu\varepsilon_{23}$$

The constitutive equation for elasticity can also be written in terms of stress:

$$\varepsilon_{ij} = \frac{\tau_{ij}}{2\mu} - \frac{\lambda\delta_{ij}}{(2\mu + 3\lambda)}\tau_{kk}$$

Equations for the normal strains are written:

$$\varepsilon_1 = \frac{1}{E}\tau_1 - \frac{\upsilon}{E}\tau_2 - \frac{\upsilon}{E}\tau_3$$

$$\varepsilon_2 = -\frac{\upsilon}{E}\tau_1 + \frac{1}{E}\tau_2 - \frac{\upsilon}{E}\tau_3$$

$$\varepsilon_3 = -\frac{\upsilon}{E}\tau_1 - \frac{\upsilon}{E}\tau_2 + \frac{1}{E}\tau_3$$

E is Young's modulus and ν is Poisson's ratio. In terms of the Lamé parameters, Young's modulus and Poisson's ratio are given by:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

$$\upsilon = \frac{\lambda}{2(\lambda + \mu)}$$

The equations for the shear strains are written:

$$\varepsilon_{12} = \frac{\tau_{12}}{2\mu}$$

$$\varepsilon_{13} = \frac{\tau_{13}}{2\mu}$$

$$\varepsilon_{32} = \frac{\tau_{32}}{2\mu}$$

2. Examples of Elastic Behavior

This section describes a few simple examples of elasticity and demonstrates how to reduce the Lamé parameters to conventional and more manageable moduli.

a. Hydrostatic compression

In hydrostatic compression, all shear stresses are zero and all normal stresses are equal:

$$\tau_{ij} = -p\delta_{ij}$$

Substituting this expression into the constitutive equation for elasticity yields:
$$\tau_{ij} = -p\delta_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$
$$-3p = (3\lambda + 2\mu)\varepsilon_{kk}$$

This equation can be rewritten in terms of volume change because ε_{kk} equals cubic dilation:

$$-3p = (3\lambda + 2\mu)\frac{\Delta V}{V}$$

$$-\frac{\Delta V}{V}p = \lambda + \frac{2}{3}\mu$$

The right-hand side of the bottom equation is called the bulk modulus and is denoted by K.

$$-\frac{\Delta V}{V}p = K$$

b. Uniaxial stress

In uniaxial stress, the only nonzero stress is a normal stress:
$$\tau_{11} = T$$
$$\text{other } \tau_{ij} = 0$$

Under these conditions, the elasticity equation written in terms of stress becomes

$$2\mu\varepsilon_{11} = T - \frac{\lambda}{(2\mu + 3\lambda)}T$$

The Lamé parameters can be grouped into a single term called Young's Modulus E so that the equation can be written as a simple proportionality:

$$T = \frac{\mu(2\mu + 3\lambda)}{(\mu + \lambda)}\varepsilon$$
$$T = E\varepsilon$$

This equation is called Hooke's Law.

c. Uniaxial strain

In uniaxial strain, the only nonzero strain is a normal strain:

$$\varepsilon_{11} = \varepsilon$$
$$other\,\varepsilon_{ij} = 0$$

Under these conditions, the elasticity equation in terms of strain becomes

$$\tau_{11} = (\lambda + 2\mu)\varepsilon$$
$$\tau_{22} = \tau_{33} = \lambda\varepsilon$$

These equations can be combined to give a relationship between only the stresses:

$$\tau_{22} = \tau_{33} = \frac{\lambda}{(\lambda + 2\mu)}\tau_{11} = \frac{\upsilon}{1-\upsilon}\tau_{11}$$

This equation is helpful for determining how sedimentation—a process that approximates uniaxial strain—affects the state of stress in the lithosphere. Consider the following picture:
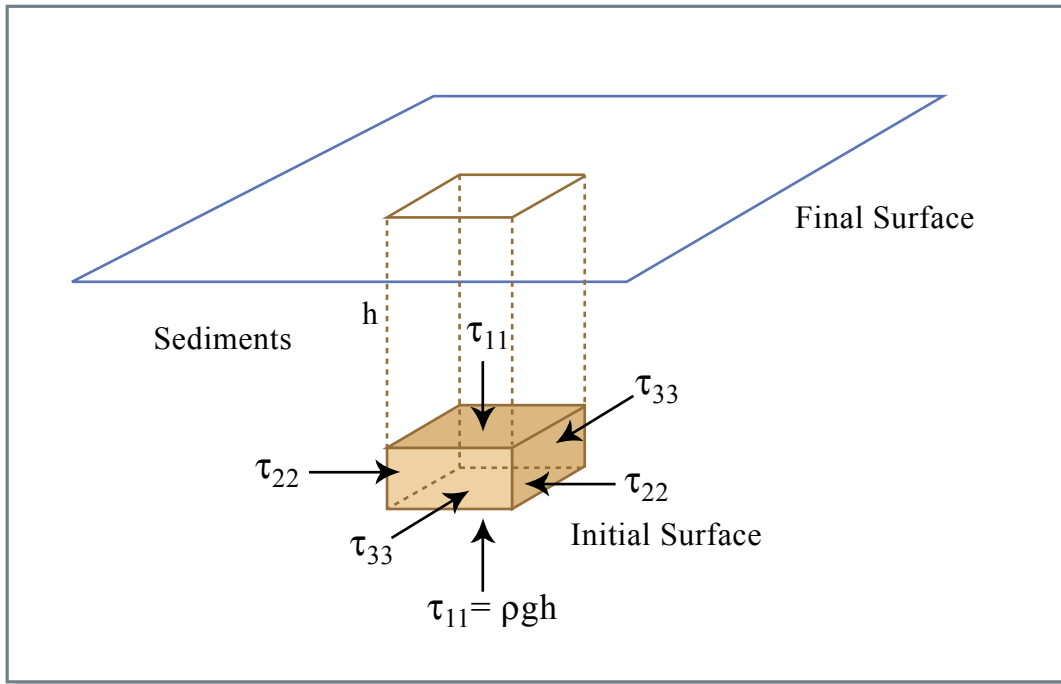

Figure 11.2

Figure by MIT OCW.

If a load of sediment with density ρ is deposited in a column h meters thick, the stress at the base of the column τ_{11} is given by:

$$\tau_{11} = -\rho g h$$

The two horizontal stresses perpendicular to the column are given by

$$\tau_{22} = \tau_{33} = -\frac{\lambda}{(\lambda + 2\mu)} \rho g h = -\frac{\upsilon}{1-\upsilon} \rho g h$$

The deviatoric stresses τ_i'are found by subtracting the pressure p from each of the stresses τ_i=1,2,3 :

$$p = \frac{1}{3}\tau_{ij}\delta_{ij} = \frac{1}{3}(-\rho g h - \frac{\upsilon}{1-\upsilon}\rho g h - \frac{\upsilon}{1-\upsilon}\rho g h) = \frac{-\rho g h}{3}\frac{1+\upsilon}{1-\upsilon}$$

$$\tau_1'= \tau_1 - p = -\rho g h - \frac{-\rho g h}{3}\frac{1+\upsilon}{1-\upsilon} = \frac{-2\rho g h(2-\upsilon)}{3(1-\upsilon)}$$

$$\tau_2'= \tau_3'= -\frac{\upsilon}{1-\upsilon}\rho g h - (\frac{-\rho g h}{3}\frac{1+\upsilon}{1-\upsilon}) = \frac{\rho g h(1-2\upsilon)}{3(1-\upsilon)}$$

The above formulas show that while the vertical stress is compressive, the horizontal stresses are tensional.