Lecture notes for 12.086/12.586, Modeling Environmental Complexity

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September 22, 2014

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1 Self-organized criticality

Scheidegger's model of rivers has been rediscovered in many different contexts.

A particularly interesting example is the directed sandpile model.

But we first discuss the "classical" undirected sandpile model of Bak, Tang, and Wiesenfeld [1].

On a grid of size $L \times L$, assign a number

$$Z(x,y) = \text{number of sand grains at location } x, y$$

At each time step, choose a site randomly and add one grain of sand:

$$Z(x,y) \to Z(x,y) + 1.$$

Continue to repeat these random additions until a site is "unstable" and topples.

The toppling rule: when $Z(x,y) \geq Z_c$,

$$Z(x,y) \to Z(x,y) - 4$$

and a grain of sand is added to each of the nearest neighboring sites:

$$
\begin{aligned}
Z(x \pm 1, y) &\rightarrow Z(x \pm 1, y) + 1 \\
Z(x, y \pm 1) &\rightarrow Z(x, y \pm 1) + 1.
\end{aligned}
$$

When sand is transferred to neighboring sites, the neighboring sites themselves can also topple. As can their neighbors, etc. The toppling continues until no more sites are unstable. If a boundary of the system is reached, particles fall off the edge.

Click here to see a simulation of the sandpile model.

(An alternative formulation, due to Grassberger [1], considers a room full of bureaucrats. When their paper work piles up beyond a critical level, they pass off one unit of work to each of their neighboring bureaucrats...)

1.1 Avalanches

The interest in such a model is in the *avalanches.* Define the avalanche size

$$ s = \text{number of sites that topple in a single event.} $$

Simulations show a power-law distribution of s:

$$ P(s) \sim s^{-\tau}, \qquad \tau \simeq 1.1 $$

The importance of such a relation is that there is no characteristic avalanche size (i.e., the distribution's mean and variance exist only if one assumes a finite system size L).

One can find data for real avalanches, or perform experiments with laboratory sandpiles, that, at least in some cases, confirm this picture.

However its importance derives not from its detailed correspondence to real sandpiles but instead the apparent ubiquity of such power-law phenomena in natural systems.

1.2 Earthquakes

Consider, for example, an *earthquake fault.* Model it on a grid, so that $Z(x, y)$ represents stress supported by a local *asperity* (i.e., friction due to surface roughness).

Regional stress can be modeled by successively adding one unit of stress at random sites. When stress exceeds a critical threshold, the asperity breaks such that 4 units of stress are removed and one unit is distributed to each of the nearest neighbors.

This model is of course merely a reinterpretation of the sandpile model. But the distribution of avalanches, reinterpreted as earthquakes, has a remarkable correspondence to the *Gutenberg-Richter scaling law*: the number N of earthquakes of magnitude m (amount of energy released) scales like

$$N(m) \sim m^{-b}, \qquad b \simeq 1$$

1.3 Critical phenomena

When the temperature of a magnet is raised to a critical temperature T_c, the magnetization M decreases sharply, with infinite slope, to zero. If held at the critical point, one finds zones of magnetization of all sizes.

Analogous phenomena occur at the critical point of fluids, where vapor and liquid phases become indistinguishable, and density fluctuations occur at all scales.

This *scale-free* behavior is characteristic of all *equilibrium critical phenomena,* and is independent of the material properties of the system.

To be more precise about scale invariance, suppose that the fluctuation size x scales like

$$p(x) \propto x^{-\alpha}.$$

Note that this distribution is invariant, except for a prefactor, under the

change of scale $x \to bx$:

$$p(bx) = (bx)^{-\alpha} = b^{-\alpha}p(x).$$

The power law is the only distribution of this type [2].

The lack of a characteristic fluctuation size is a general property of *equilibrium critical phenomena*, wherein long-range correlations exist at the critical temperature T_c.

Bak's sandpile model is interesting because it exhibits such critical behavior without any "tuning" of the temperature. The system instead "self-organizes" to the critical state. Consequently the phenomenon is known as *self-organized criticality*.

1.4   Directed sandpiles

We proceed to show how Scheidegger's rivers relate to Bak's avalanches.

We make one key change to Bak's sandpile: we assume that the "tilt" of the pile requires that sand flow only downhill [3, 4].

We tilt a square lattice diagonally, and add particles at random sites on the top row. We take $Z_c = 2$, so that when $Z(x, y) \geq Z_c$,

$$Z(x, y) \to Z(x, y) - 2$$

and we add "sand" to the neighboring sites in the nearest lower row:

$$Z(x \pm 1, y - 1) \to Z(x \pm 1, y - 1) + 1$$

If those sites are now unstable, the sand is transferred to the nearest neighbors below once again, and so on.

The avalanching continues until all sites are once again stable such that $Z \leq Z_c = 2$ everywhere.

The correspondence with the Scheidegger model follows from identifying the paths taken by unstable particles of sand with those of Scheidegger's raindrops or random walkers.

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Specifically, we once again see that the avalanche "basin" is bounded by directed random walks, and we inherit not only the same first-passage problem of Scheidegger but also the same exponents.

In particular, we have the avalanche size distribution

$$P(s) \sim s^{-\tau}, \qquad \tau = 4/3.$$

Finally, we note that Takayasu's model of random aggregation [5] is also equivalent to Scheidegger's model. In this case particles move about randomly on a 1D lattice, with constant reinjection of mass. In steady state, the distribution of mass is the same as the basin distribution or the avalanche distribution.

1.5   The lessons learned

There are two principle lessons:

- The allometry imposed by random walks can account for a wide variety of apparently unrelated phenomena in which size distributions decay as power laws.

- The lack of a characteristic fluctuation size—i.e., scale invariance—can occur *without* the need to "tune" a control parameter to a critical value, as in equilibrium critical phenomena.

The first lesson does does not in itself mean that rivers, sandpiles, etc. are truly the result of some mechanism that generates random walks. Instead it merely establishes consistency with the geometry of random walks.

The second lesson is deeper, in that it suggests that the critical state is not so special. Indeed we shall see an example later, when we discuss percolation theory, where this issue displays itself in a surprising and powerful way in a dynamical context.

References

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[5] Takayasu, H., Nishikawa, I. & Tasaki, H. Power-law mass distribution of aggregation systems with injection. *Physical Review A* **37**, 3110–3117 (1988).

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Fall 2014