12.003 Atmosphere, Ocean and Climate Dynamics
Fall 2008

Problem set 9: Ocean Circulation

Due date: December 5th, 2008

1. Use Sverdrup theory and the idea that only western boundary currents are allowed, to sketch the pattern of ocean currents you would expect to observe in the basin sketched in the figure below in which there is an island. Assume a wind pattern of the form sketched in the figure.

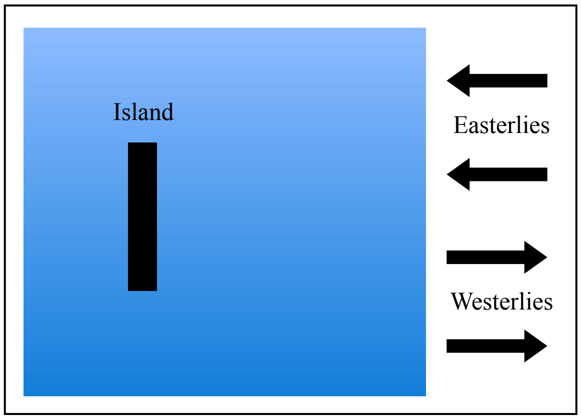

Figure 1: The figure depicts an ocean basin with an island in the middle. North is up and West is right. The wind pattern blowing on the ocean is sketched on the left of the figure.

2. In class we showed that the wind-driven ocean circulation can be inferred from knowledge of the surface wind stress. A shortcoming of the derivation is that it did not predict the emergence of western boundary currents; the theory only applied to the broad flows in the gyre interior. Henry Stommel, a star in the field of oceanography, showed that the emergence of western boundary currents can be predicted, if one accounts for bottom friction. You are here given the opportunity of becoming a star yourself and repeat Stommel's reasoning.

   Consider an ocean in a rectangular domain $0 \leq y \leq L$, $0 \leq x \leq W$ of constant depth H filled with water of constant density, i.e. $\rho = \rho_{ref} = \text{const}$. The ocean dynamics at large scales in such an ocean is governed by the equations,

$$-fv = -\frac{1}{\rho_{ref}}\frac{\partial p}{\partial x} + \frac{1}{\rho_{ref}}\frac{\partial \tau^x}{\partial z}, \tag{1}$$

$$+fu = -\frac{1}{\rho_{ref}}\frac{\partial p}{\partial y} + \frac{1}{\rho_{ref}}\frac{\partial \tau^y}{\partial z}, \tag{2}$$

$$0 = -\frac{1}{\rho_{ref}}\frac{\partial p}{\partial z} - g, \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

Assume that at the ocean surface $\tau^{(x)} = -\tau_0 \cos(\pi y/L)$ and $\tau^{(y)} = 0$, while at the bottom $\boldsymbol{\tau} = -r\boldsymbol{U}$, where r is the friction coefficient and $\boldsymbol{U}$ is the vertically integrated horizontal velocity. The addition of a bottom stress is the only addition to the Sverdrup's problem described in class. It will turn out to be crucial to allow western boundary currents.

(a) Subtract the y-derivative of (1) from the x-derivative of (2). This is the equation for the ocean vorticity $\zeta = \partial_y v - \partial_x u$.

(b) Calculate the vertical integral of the vorticity equation. You will need to use the expression for the surface and bottom stresses here.

(c) Using the continuity equation, show that the vertically integrated velocity field is divergenceless, i.e. $\partial_x U + \partial_y V = 0$.

(d) A divergenceless velocity field can always be expressed in terms of a streamfunction ψ such that $(U, V) = (-\partial_y \psi, \partial_x \psi)$. Show that the vertical integral of the vorticity equation can be expressed as an equation for the streamfunction ψ:

$$\beta \frac{\partial \psi}{\partial x} + r\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = \frac{\tau_0 \pi}{\rho_{ref} L} \sin\left(\frac{\pi y}{L}\right). \tag{5}$$

(e) Verify that the solution of the equation for ψ is:

$$\psi = \frac{\tau_0 L}{\pi \rho_{ref} r} \Psi(x) \sin\left(\frac{\pi y}{L}\right)$$

where

$$\Psi(x) = \frac{1 - e^{bW}}{e^{aW} - e^{bW}} e^{ax} - \frac{1 - e^{aW}}{e^{aW} - e^{bW}} e^{bx} - 1$$

and

$$a = -\frac{\beta}{2r} + \sqrt{\frac{\beta^2}{4r^2} + \frac{\pi^2}{L^2}}, \qquad b = -\frac{\beta}{2r} - \sqrt{\frac{\beta^2}{4r^2} + \frac{\pi^2}{L^2}}.$$

Check also that ψ vanishes at $x = 0$ and $x = W$. Why is this important?

(f) Make contour plots of ψ for two different values for $r = 10^{-6}\text{s}^{-1}$ (weak bottom friction) and $r = 10^{-4}\text{s}^{-1}$ (strong bottom friction). What is the maximum value of ψ in each case? Do you see a western boundary current in the two solutions? How does it depend on r? [For this exercise use $\tau_0 = 0.1\text{N m}^{-2}$, $L = 6{,}000\text{km}$, $W = 10{,}000\text{km}$.]

(g) Make contour plots of the dissipation term $r\partial^2\psi/\partial x^2 + r\partial^2\psi/\partial x^2$ for the two values of r. Where is the dissipation largest? What term balances dissipation in the vorticity equation (5) in each case?

3. The figure below shows the distribution of salinity in the Indian Ocean along 80°E from India to Antarctica. On the figure, draw the mid-line of at least one water mass, and sketch the possible flow of water in the core as indicated by the distribution of salinity.

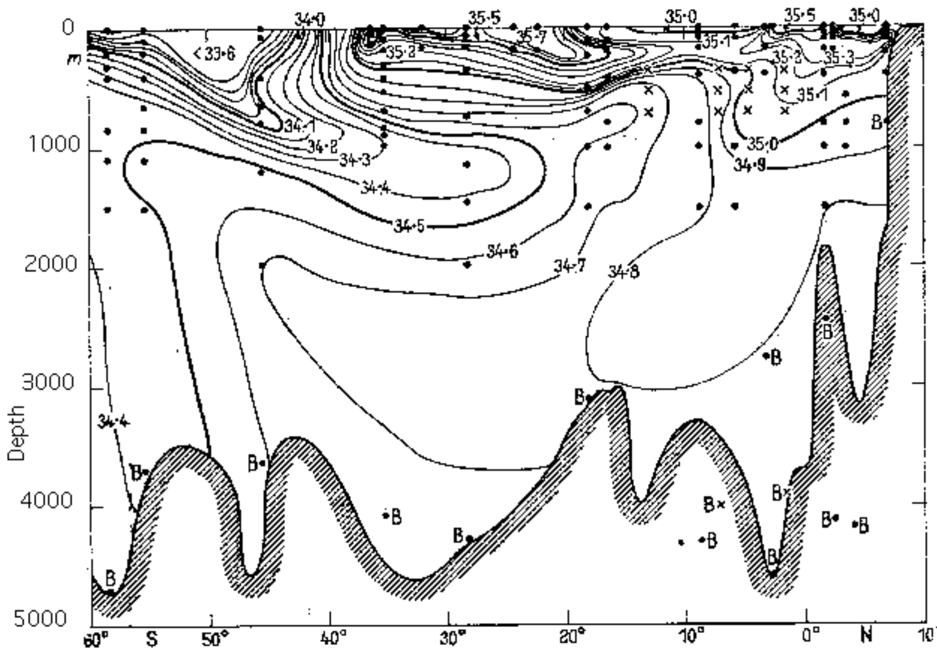

Figure 2: Distribution of salinity in the Indian Ocean along 80°E.