Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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1.4 Forces in the Fluid

There are two types of forces acting on a fluid element:

1. Short-range force (surface force): It is molecular in origin, and decreases rapidly with the distance between interacting elements. This force is appreciable only if fluid elements are in contact, therefore exists only on the boundary, and is called a "surface force." Surface force per unit area $\vec{\Sigma}$ is called the stress, which depends on time, on the location, $\vec{x}$, and on the orientation of the surface element, i.e., or its unit normal $\vec{n}$.

2. Long-range force (volume force, body force): The origin of the force is far away from the zone of interest. The strength of such forces varies very slowly, and acts uniformly on all parts of a fluid parcel. Therefore, the total force is proportional to the volume of fluid. We define $\vec{f}$ to be the "body force" acting on a unit volume, then the "body force" on a given mass ρdV is $\rho \vec{f} dV$. A typical body force is gravity.

1.4.1 Surface force and stresses

Consider a cubic element, depicted in Figure 1.4.1. On any surface element there are three components of stresses $\vec{\Sigma}$. We denote the component in the direction of j acting on the surface element whose normal is in i direction by σ_{ij}, thus the stress components form a square array:

$$\{\sigma_{ij}\} = \left\{ \begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array} \right\} \tag{1.4.1}$$

The first subscript indicates the direction of the unit normal to the surface element, and the second subscript indicates the direction of the stress component. The entire array $\{\sigma_{ij}\}$ is called the stress tensor (see notes on Tensors).

The diagonal terms $\sigma_{ii} \quad i = 1, 2, 3$, are the normal stress components; the off-diagonal terms $\sigma_{ij} \quad i \neq j \quad 1, 2, 3$, are the shear stress components.

We shall first show that the stress tensor is symmetric

$$\sigma_{ij} = \sigma_{ji}, \tag{1.4.2}$$

Let the length of each side of a cubic element be $\Delta\ell$. Conservation of angular momentum requires :

$$I \frac{d\omega}{dt} = -\sigma_{12} \Delta\ell^2 \cdot \Delta\ell + \sigma_{21} \Delta\ell^2 \Delta\ell$$

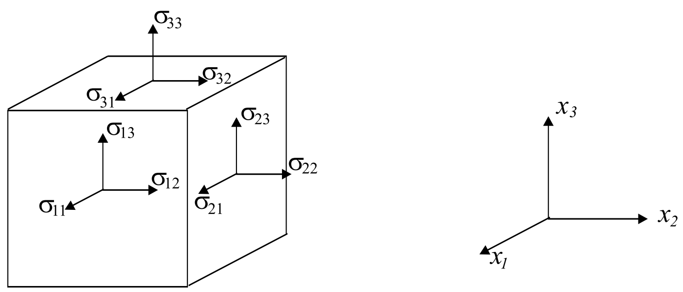

Figure 1.4.1: Stress components on a fluid element

From left to right, the terms represent, respectively, the angular inertia, the torque due to shear stress on the two vertical surfaces and the torque due to shear on two horizontal surfaces. Now the moment of inertia $I \propto \rho(\Delta\ell)^5$. Hence, as $\Delta\ell \to 0, \sigma_{12} = \sigma_{21}$ as long as $d\omega/dt \neq \infty$. After similar arguments for all other off-diagonal components, we prove (1.4.2). Thus among nine components, only six can be distinct.

1.4.2 Cauchy's theorem

Are σ_{ij}, defined on three mutually orthogonal surfaces in a chosen coordinate system, capable of describing stresses on any surface? In other words, can the stress components on any surface be expressed in terms of σ_{ij} ? To answer this question let us consider a tetrahedron shown in Figure 1.4.2. Three sides are formed by orthogonal surface elements $dS_1 = BCO, dS_2 = ACO, dS_3 = ABO$ whose unit normals are $-\vec{e}_1, -\vec{e}_2, -\vec{e}_3$, respectively. The fourth side $dS = ABC$ is inclined with the unit normal $\vec{n} = (n_1, n_2, n_3)$ pointing in an arbitrary direction.

First, let's show that

$$\begin{aligned} dS_1 &= n_1 dS = \cos(\vec{n}, \vec{e}_1)\, dS \\ dS_2 &= n_2 dS = \cos(\vec{n}, \vec{e}_2)\, dS \\ dS_3 &= n_3 dS = \cos(\vec{n}, \vec{e}_3)\, dS \end{aligned}$$

In other words

$$dS_i = n_i dS, \quad i = 1, 2, 3. \tag{1.4.3}$$

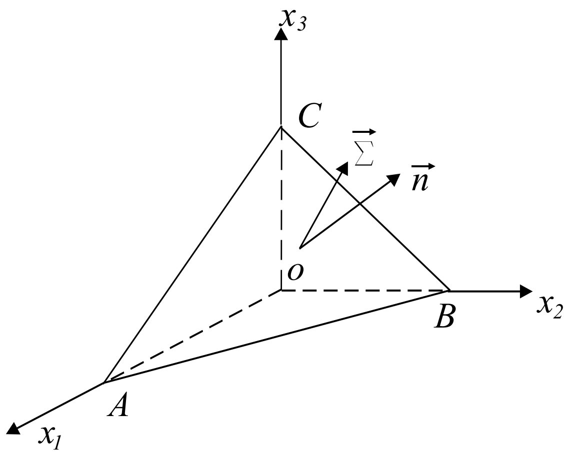

Figure 1.4.2: Stress components on any surface element

Consider an *arbitrary constant* vector $\vec{A}$. By Gauss' theorem

$$\iiint_V \nabla \cdot \vec{A} dS = \iint_S \vec{A} \cdot \vec{n} dS,$$

where V is the volume of the tetrahedron and S the total surface of tetrahedron. Since $\vec{A}$ is a constant vector, $\nabla \cdot \vec{A} = 0$.

$$0 = \iint_S \vec{A} \cdot \vec{n} dS = \vec{A} \cdot \iint_S \vec{n} dS = 0.$$

Since $\vec{A}$ is arbitrary

$$\iint_S \vec{n} dS = 0 = \vec{n} dS - \vec{e}_1 dS_1 - \vec{e}_2 dS_2 - \vec{e}_3 dS_3.$$

Hence

$$\begin{aligned}
\vec{n} \cdot \vec{e}_1 dS &= n_1 dS = dS_1 \\
\vec{n} \cdot \vec{e}_2 dS &= n_2 dS = dS_2 \\
\vec{n} \cdot \vec{e}_3 dS &= n_3 dS = dS_3
\end{aligned}$$

Note that the volume of the tetrahedron is $\frac{1}{3}hdS$ where h is the vertical distance from the origin to the surface dS.

Next we consider the force balance in the x_1 direction, i.e., $F_x = ma_x$:

$$-\sigma_{11}n_1 dS - \sigma_{21}n_2 dS - \sigma_{31}n_3 dS + \Sigma_1 dS + \rho f_x \frac{h}{3} dS = \rho \frac{h}{3} dS \frac{dq_1}{dt}. \tag{1.4.4}$$

where f_x is the x component of the body force per unit volume. As the tetrahedron shrinks to a point $h \to 0$, the last two terms diminish much faster than the rest by a factor h for any finite f_i and dq_1/dt, $i = 1, 2, 3$, hence,

$$\begin{aligned} \Sigma_1 &= \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3 \\ &= \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 \qquad (\sigma_{21} = \sigma_{12}) \\ &= \sigma_{1j}n_j. \end{aligned}$$

Similar results are obtained by considering the force balance in two other directions. In summary, we have

$$\Sigma_i = \sigma_{ij}n_j \tag{1.4.5}$$

which states that the stress $\vec{\Sigma}$ on a surface element of any orientation is a linear superposition of the stress components defined on a cube in some Cartesian coordinate system. This result is called Cauchy's formula.

Note that Σ_i is tensor of rank 1 and $\vec{n} = \{n_j\}$ is an arbitrary tensor of rank 1 (because the surface dS is arbitrary). Equation (1.4.5) states that the scalar product of the array of numbers σ_{ij} with an arbitrary tensor n_j of rank 1 is another tensor Σ_i of rank 1. By the quotient law the array $\{\sigma_{ij}\}$ must be a tensor of rank two.