1.225J (ESD 205) Transportation Flow Systems

Lecture 5

Assignment on Traffic Networks

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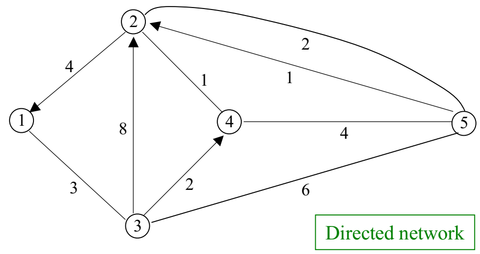
Lecture 5 Outline

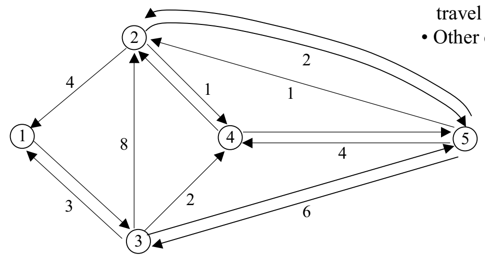
- Summary from previous lectures:
  - Assignment on non-congested networks: All-or-nothing assignment
  - Volume-delay functions for "congested" networks
- Framework for static traffic assignment models
- Static traffic assignment: concepts
- Static traffic assignment: principles
- User Optimal (UO) and System Optimal (SO) static traffic assignment
- Summary

Non-Congested Road Network and O-D Matrix

Mixed network



Directed network



O-D Matrix

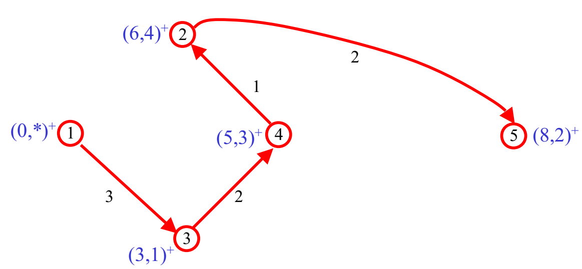
$[t(i,j)] =$

From \ To	1	2	3	4	5
1	–	30	35	40	15
2	10	–	15	12	10
3	50	40	–	35	20
4	25	30	35	–	40
5	45	30	35	40	–

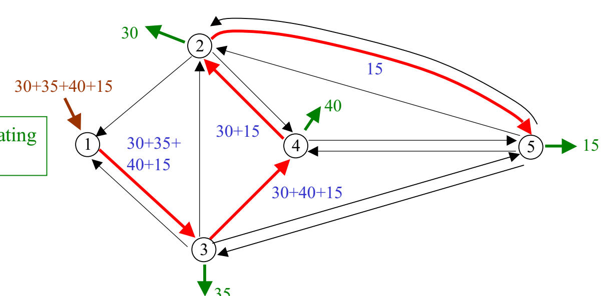
- In R7, $t(i,j)$ means demand for O-D pair (i,j). ($t(i,j)$ does not mean travel time of link *(i,j)*
- Other common notation: $q(i,j)$ or q_{ij}

Assign O-D Flows Originating from Node 1

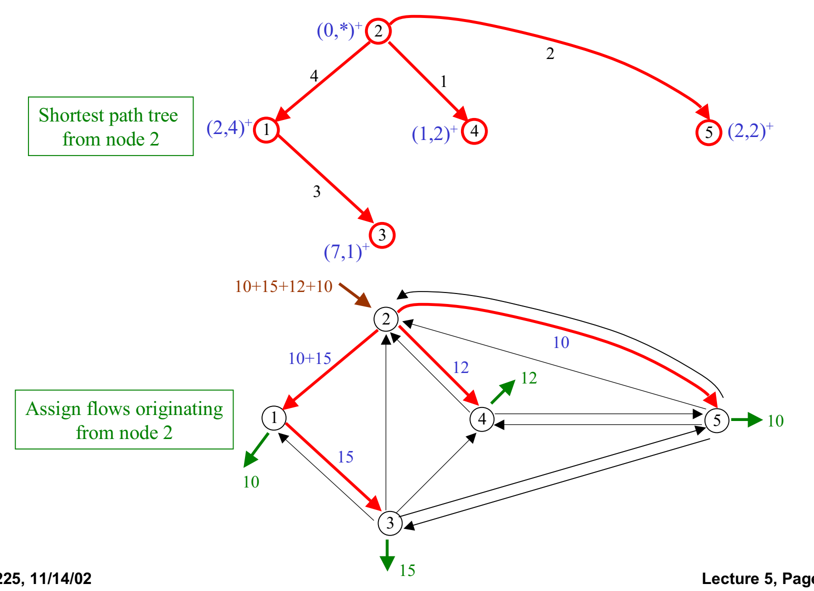
Shortest path tree from node 1



Assign flows originating from node 1



Assign O-D Flows Originating from Node 2



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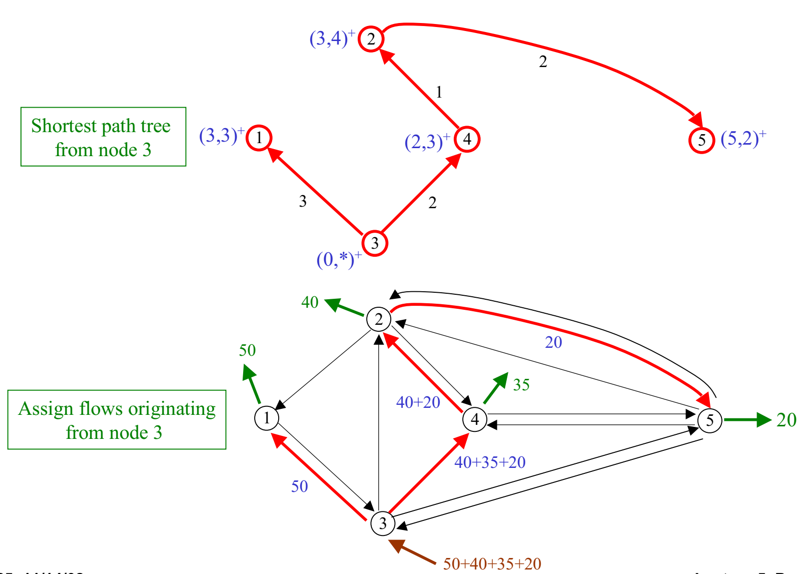
Assign O-D Flows Originating from Node 3



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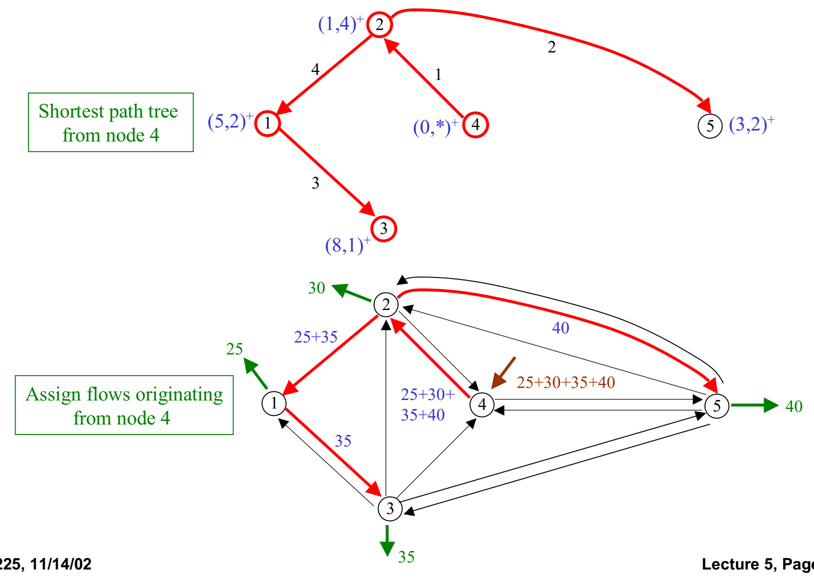
Assign O-D Flows Originating from Node 4



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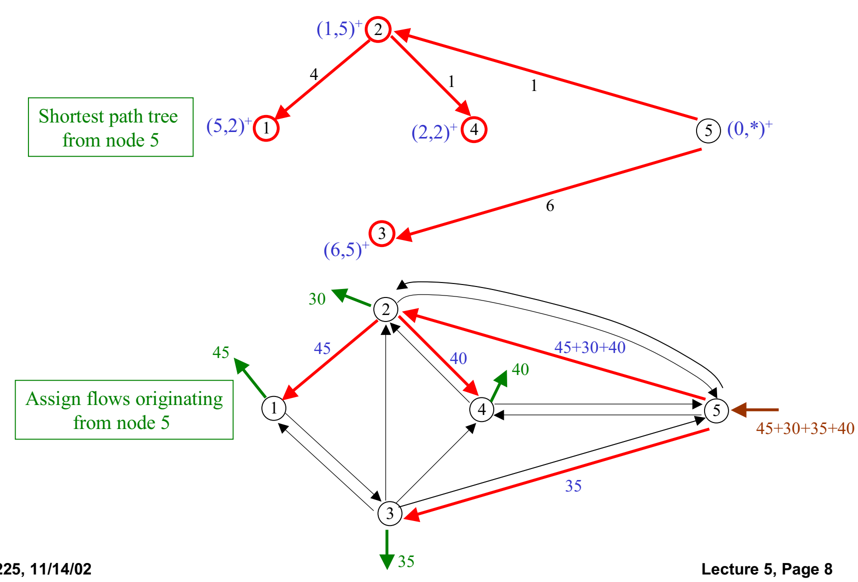
Assign O-D Flows Originating from Node 5



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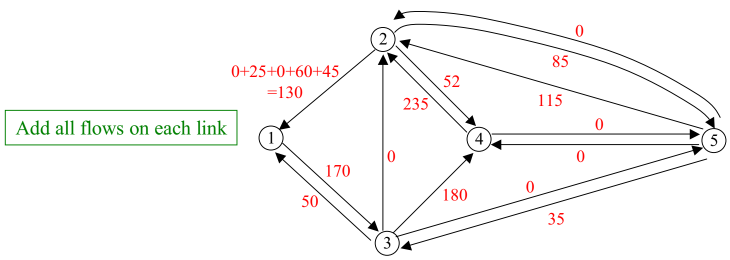
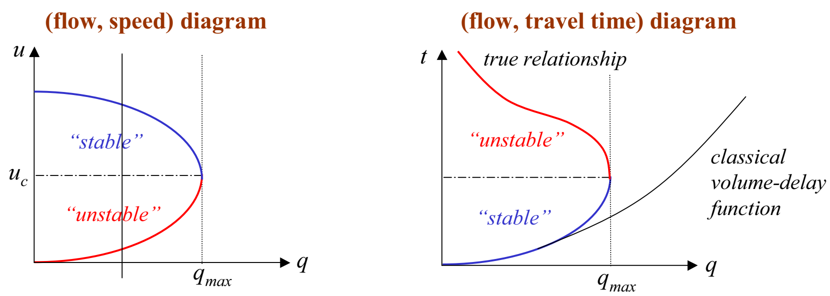
Results of All-or-Nothing Assignment



- ***All-or-nothing (AON) assignment*** does not consider congestion
- The solution may not be unique (Why? Is unique solution important?
- AON assignment does not make sense if certain links are congested
- To account for congestion, travel time must depend on link flow
- How to change the AON assignment method to make it work?

  ⇒ ***Equilibrium traffic assignment***

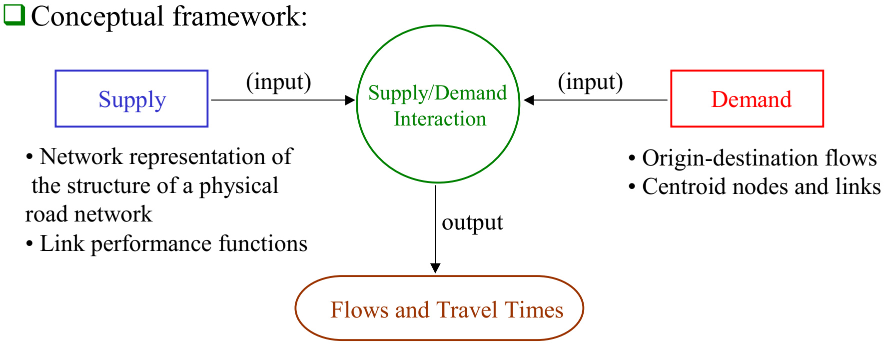
Derived Diagrams from the Fundamental Diagram



- In general, q cannot be used as a variable (why?)
- In the traffic planning area:
  - q is also called **volume**
  - travel time is also called **travel delay**
  - In the case of volume-delay functions, q is used as a variable

Framework for Static Traffic Assignment Models

- Conceptual framework:



- Principles of assignment to represent the supply/demand interaction
  - *User Optimal (U.O.)*: O-D flows are assigned to paths with minimum travel time
  - *System Optimal (S.O.)*: O-D flows are assigned such that total travel time on the network is minimum

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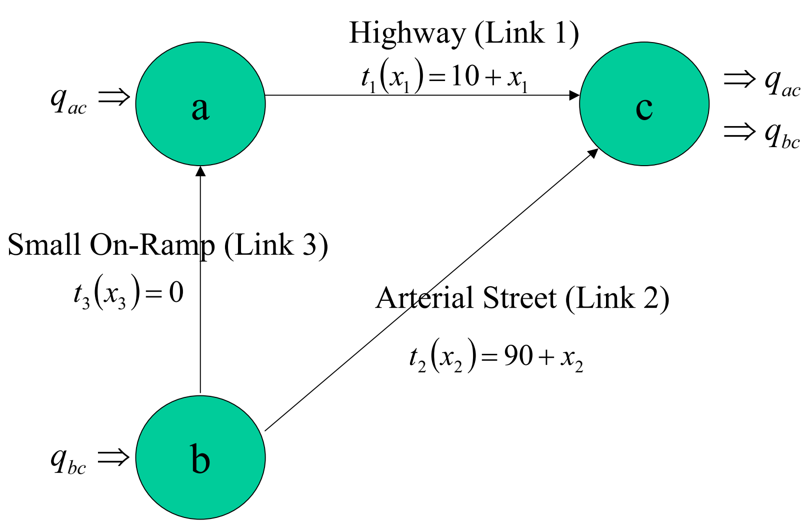
Example



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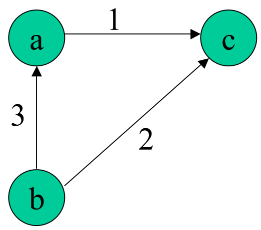
Traffic Assignment Concepts

- Conceptual Network



- Demand
  - O - Ds : (a,c) and (b,c)
  - (a,c) : q_{ac} *vehicles / hr*
  - (b,c) : q_{bc} *vehicles / hr*

- Path flow variables
  - O - D (a,c) : one path
    1 : Link 1 (f_1^{ac})
  - O - D (b,c) : two paths
    1 : Link 2 (f_1^{bc})
    2 : Link 3, Link 1 (f_2^{bc})

- O-D flows and path flows

$$q_{ac} = f_1^{ac}$$

$$q_{bc} = f_1^{bc} + f_2^{bc}$$

Traffic Assignment Concepts (cont.)-1

- Link (arc) flows and path flows

$$x_1 = f_1^{ac} + f_2^{bc}$$

$$x_2 = f_1^{bc}$$

$$x_3 = f_2^{bc}$$

- **Arc-path incidence matrix**

		a - c	b - c	
O-D →		a - c	b - c	
Path →		1	1	2
Link	1	1	0	1
	2	0	1	0
	3	0	0	1

- Assume that $f_2^{bc} = pq_{bc}$

$$x_1 = q_{ac} + pq_{bc}$$

$$x_2 = (1-p)q_{bc}$$

$$x_3 = pq_{bc}$$

- **Assignment matrix**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & p \\ 0 & 1-p \\ 0 & p \end{bmatrix} \begin{bmatrix} q_{ac} \\ q_{bc} \end{bmatrix}$$

Traffic Assignment Concepts (cont.)-2

- t_1, t_2, t_3 are the travel times of links 1, 2, 3.
- "Congested" networks:
  - Link travel times depend on link flows
  - Example:

    $t_1(x_1) = 10 + x_1,\ t_2(x_2) = 90 + x_2,\ t_3(x_3) = 0$
- Path travel-times as a function of link travel-times:

  $C_1^{ac} = t_1,\ C_1^{bc} = t_2,\ C_2^{bc} = t_1 + t_3$
- Total travel times (What is the unit of this quantity?):

  $x_1 * (10 + x_1) + x_2 * (90 + x_2) + x_3 * (0)$

Assignment Principles

- If OD flows are infinitely divisible:
  - There is an infinite number of assignments
  - Which assignment should we choose?

    $\Rightarrow$ We need additional assumptions to define a less ambiguous assignment
- Assignment principle: a principle used to determine an assignment
- Examples of assignment principles:
  - **User-optimal**: between each O-D pair, all used paths have equal and minimum travel times
  - **System-optimal**: the total travel times are minimum

Mathematical Expressions of Assignment Principles

- User-optimal traffic assignment principle: find p such that:
  - O-D (b, c):

$$\begin{array}{ll} If\ p=0, & (t_1+t_3)>=t_2 \\ If\ p=1, & (t_1+t_3)<=t_2 \\ If\ 0<p<1, & t_1+t_3=t_2 \end{array}$$

  - O-D (a, c): the question is not posed as there is only one path
- System optimal: find p that minimizes:

$$f_1^{ac}C_1^{ac}+f_1^{bc}C_1^{bc}+f_2^{ac}C_2^{ac}\ (=x_1t_1+x_2t_2+x_3t_3)$$

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Solution of the U.O. Assignment

- We want to solve for a $p\in[0,1]$ such that:

$$\begin{array}{ll} If\ p=0, & (t_1(x_1)+t_3(x_3))>=t_2(x_2) \\ If\ p=1, & (t_1(x_1)+t_3(x_3))<=t_2(x_2) \\ If\ 0<p<1, & t_1(x_1)+t_3(x_3)=t_2(x_2) \end{array}$$

and

$$q_{ac}(=80) \qquad q_{bc}(=10)$$

$$t_1(x_1)=10+x_1 \qquad t_2(x_2)=90+x_2 \qquad t_3(x_3)=0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 & p \\ 0 & 1-p \\ 0 & p \end{bmatrix}\begin{bmatrix} q_{ac} \\ q_{bc} \end{bmatrix}$$

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Solution of the U.O. Assignment (cont.)

- $t_2(x_2) - (t_1(x_1) + t_3(x_3)) = (90 + x_2) - (10 + x_1) = 80 + x_2 - x_1$
- $x_2 - x_1 = (q_{bc} - x_3) - (q_{ac} + x_3) = (q_{bc} - q_{ac}) - 2x_3$
- $t_2(x_2) - (t_1(x_1) + t_3(x_3)) = 80 + q_{bc} - q_{ac} - 2x_3$

  $$\text{If } \frac{80 + q_{bc} - q_{ac}}{2} \geq 0, \text{ then } x_3 = \frac{80 + q_{bc} - q_{ac}}{2}$$

- Example: $(q_{ac}, q_{bc}) = (80,10) \rightarrow (x_1, x_2, x_3) = (85,5,5)$
- The travel time on any path is: 95
- Total travel time (per hour): (80+10)*95=8550 veh-hrs (per hour)

(UO) Assignment May Depend on the Demand

- $\text{If } \frac{80 + q_{bc} - q_{ac}}{2} \geq 0, \text{ then}$

  $$(x_1^*, x_2^*, x_3^*) = \left(\frac{80 + q_{bc} + q_{ac}}{2}, \frac{q_{bc} - 80 + q_{ac}}{2}, \frac{80 + q_{bc} - q_{ac}}{2}\right)$$

- $\text{If } \frac{80 + q_{bc} - q_{ac}}{2} < 0, \text{ then } (x_1^*, x_2^*, x_3^*) = (q_{ac}, q_{bc}, 0)$

  $$\text{Proof}: p^* = 0 \text{ and } t_2(q_{bc}) - (t_1(q_{ac}) - t_3(0)) = \frac{80 + q_{bc} - q_{ac}}{2} < 0$$

- The U.O. assignment is a function of the demand, and the function may be non-linear. Example:

  $$(q_{ac}, q_{bc}) = (80,10) \rightarrow (x_1^*, x_2^*, x_3^*) = (85,5,5)$$
  $$(q_{ac}, q_{bc}) = (160,20) \rightarrow (x_1^*, x_2^*, x_3^*) = (160,20,0)$$
  $$(160,20) = 2 \times (80,10) \text{ But } (160,20,0) \neq 2 \times (85,5,5)$$

- Remark: An increase in demand may lead to a decrease in the flow on a link

Building More Roads Is Not Always Better

- Without Link 3, there is only one possible assignment
  $\left(x_1^*, x_2^*\right) = \left(q_{ac}, q_{bc}\right) = (80,10)$
  Total travel times in one hour: 80*(10+80)+10*(90+10)=8200 veh-hr (in one hour)
- U.O. with Link 3:
  Total travel times (in one hour): 8550 veh-hr (in one hour)
- System travel times are worse if one adds Link 3! (Is this intuitive?)
- This phenomenon is known as Braess "Paradox", and is not an isolated phenomenon.

System Optimal Assignment

- $$\min \quad x_1 t_1(x_1) + x_2 t_2(x_2) + x_3 t_3(x_3)$$

$$s.t. \quad f_1^{ac} = q_{ac} \qquad \text{Demand}$$
$$f_1^{bc} + f_2^{bc} = q_{bc}$$

$$f_1^{ac} \geq 0, f_1^{bc} \geq 0, f_2^{bc} \geq 0 \qquad \text{Non-negativity}$$

$$x_1 = f_1^{ac} + f_2^{bc}$$
$$x_2 = f_1^{bc} \qquad \text{Definition of link flows}$$
$$x_3 = f_2^{bc}$$

- S.O. solution: $\left(x_1^*, x_2^*, x_3^*\right) = (80, 10, 0)$, if $\left(q_{ac}, q_{bc}\right) = (80, 10)$
- $\min x_1 t_1(x_1) + x_2 t_2(x_2) + x_3 t_3(x_3) = \min \int_0^{x_1} m_1(x_1) dx_1 + \int_0^{x_2} m_2(x_2) dx_2 + \int_0^{x_3} m_3(x_3) dx_3$

$$Where: \ m_1(x_1) = \frac{d(x_1 t_1(x_1))}{dx_1}, \ m_2(x_2) = \frac{d(x_2 t_2(x_2))}{dx_2}, \ m_3(x_3) = \frac{d(x_3 t_3(x_3))}{dx_3}$$

Lecture 5 Summary

- Assignment on non-congested networks: All-or-nothing assignment
- Volume-delay functions for “congested” networks and static demand
- Framework for static traffic assignment models
- Static traffic assignment concepts and principles
- User Optimal and System Optimal static traffic assignment
- Summary