Queueing Systems: Lecture 1

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October 4, 2006

Announcements

- PS #3 out this afternoon
- Due: October 19 (graded by 10/23)
- Office hours – Odoni: Mon. 2:30-4:30
  - Wed. 2:30-4:30 on Oct. 18 (No office hrs 10/16)
  - Or send me a message
- Quiz #1: October 25, open book, in class
- Old quiz problems and solutions: posted on 10/19

Topics in Queueing Theory

- Introduction to Queues
- Little's Law
- Markovian Birth-and-Death Queues
- The M/M/1 and Other Markovian Variations
- The M/G/1 Queue and Extensions
- Priority Queues
- Some Useful Bounds
- Congestion Pricing
- Queueing Networks; State Representations
- Dynamic Behavior of Queues

Lecture Outline

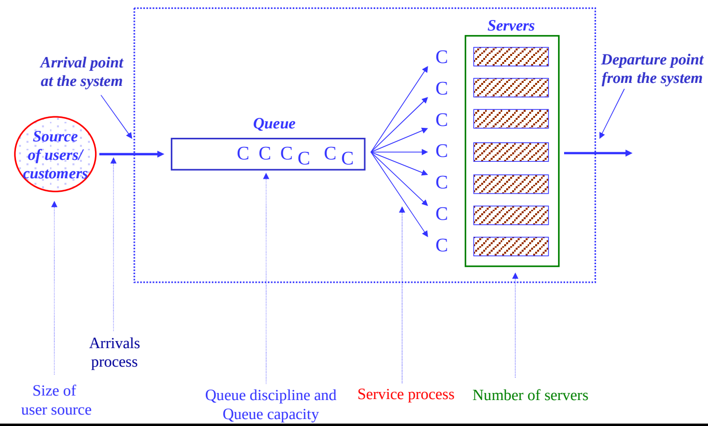
- Introduction to queueing systems
- Conceptual representation of queueing systems
- Codes for queueing models
- Terminology and notation
- Little's Law and basic relationships
- Birth-and-death models
- The M/M/1 queueing system

Reference: Chapter 4, pp. 182-203

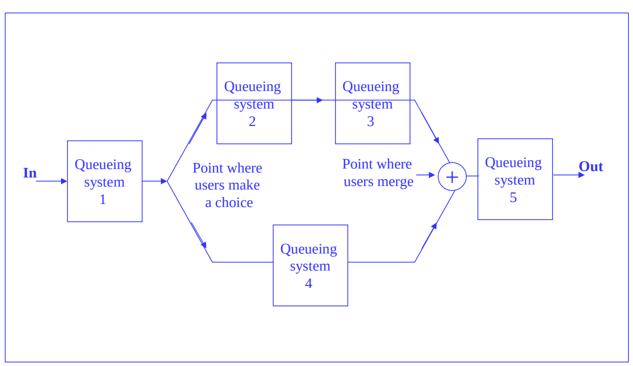
Queues

- Queueing theory is the branch of operations research concerned with waiting lines (delays/congestion)
- A queueing system consists of a user source, a queue and a service facility with one or more identical parallel servers
- A queueing network is a set of interconnected queueing systems
- Fundamental parameters of a queueing system:
  - Demand rate
  - Capacity (service rate)
  - Demand inter-arrival times
  - Service times
  - Queue capacity and discipline (finite vs. infinite; FIFO/FCFS, SIRO, LIFO, priorities)
  - Myriad details (feedback effects, “balking”, “jockeying”, etc.)

A Generic Queueing System



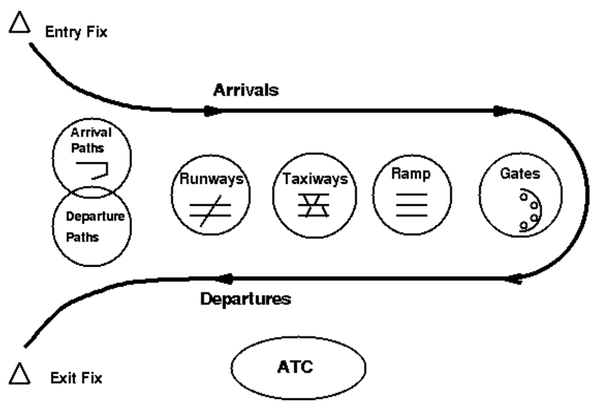
Queueing network consisting of five queueing systems



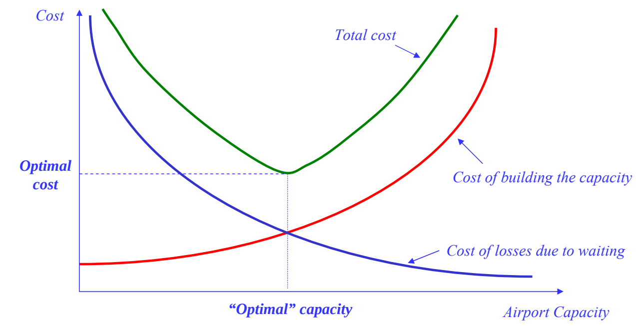
Applications of Queueing Theory

- **Some familiar queues:**
  - _ Airport check-in; aircraft in a holding pattern
  - _ Automated Teller Machines (ATMs)
  - _ Fast food restaurants
  - _ Phone center’s lines
  - _ Urban intersection
  - _ Toll booths
  - _ Spatially distributed urban systems and services
- **Level-of-service (LOS) standards**
- **Economic analyses involving trade-offs among operating costs, capital investments and LOS**
- **Congestion pricing**

The Airside as a Queueing Network



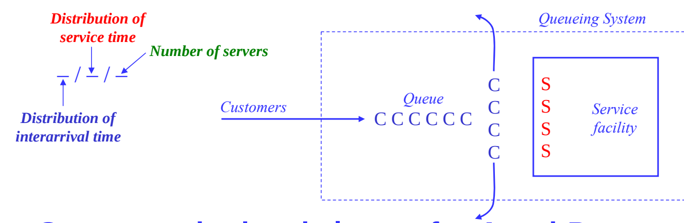
Queueing Models Can Be Essential in Analysis of Capital Investments



Strengths and Weaknesses of Queueing Theory

- Queueing models necessarily involve approximations and simplification of reality
- Results give a sense of order of magnitude, changes relative to a baseline, promising directions in which to move
- Closed-form results essentially limited to "steady state" conditions and derived primarily (but not solely) for birth-and-death systems and "phase" systems
- Some useful bounds for more general systems at steady state
- Numerical solutions increasingly viable for dynamic systems
- Huge number of important applications

A Code for Queueing Models: *A*/*B*/*m*



- **Some standard code letters for *A* and *B*:**
  - *M: Negative exponential (M stands for memoryless)*
  - *D: Deterministic*
  - *E_k:kth-order Erlang distribution*
  - *G: General distribution*

Terminology and Notation

- *Number in system*: number of customers in queueing system
- *Number in queue* or "*Queue length*": number of customers waiting for service
- *Total time in system* and *waiting time*
- $N(t)$ = number of customers in queueing system at time t
- $P_n(t)$ = probability that $N(t)$ is equal to n at time t
- λ_n: mean arrival rate of new customers when $N(t) = n$
- μ_n: mean (total) service rate when $N(t) = n$

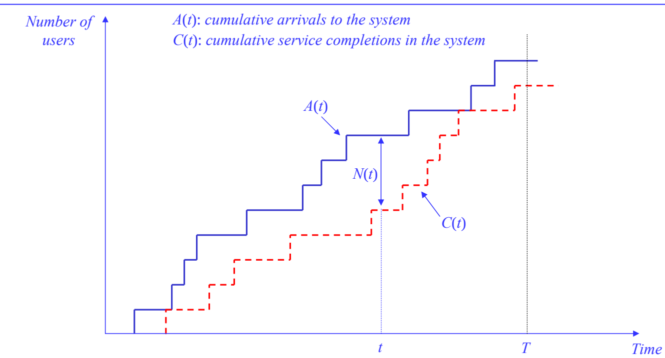
Terminology and Notation (2)

- *Transient state*: state of system at t is influenced by the state of the system at $t = 0$
- *Steady state*: state of the system is independent of initial state of the system
- m: number of servers (parallel service channels)
- If λ_n and the service rate per busy server are constants λ and μ, respectively, then $\lambda_n=\lambda$, $\mu_n= \min(n\mu, m\mu)$; in that case:
  - Expected inter-arrival time = $1/\lambda$
  - Expected service time = $1/\mu$

Some Expected Values of Interest at Steady State

- *Given:*
  - λ = arrival rate
  - μ = service rate per service channel
- *Unknowns:*
  - L = expected number of users in queueing system
  - L_q = expected number of users in queue
  - W = expected time in queueing system per user ($W = E(w)$)
  - W_q = expected waiting time in queue per user ($W_q = E(w_q)$)
- 4 unknowns $\Rightarrow$ We need 4 equations

Little's Law



$$L_T = \frac{\int_0^T N(t)dt}{T} = \frac{A(T)}{T} \cdot \frac{\int_0^T N(t)dt}{A(T)} = \lambda_T \cdot W_T$$

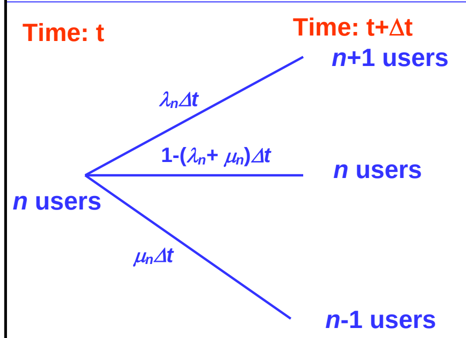
Relationships among L, L_q, W, W_q

- Four unknowns: L, W, L_q, W_q
- Need 4 equations. We have the following 3 equations:
  - $L = \lambda W$ (Little's law)
  - $L_q = \lambda W_q$
  - $W = W_q + \frac{1}{\mu}$
- If we can find any one of the four expected values, we can determine the three others
- The determination of L (or other) may be hard or easy depending on the type of queueing system at hand
- $L = \sum_{n=0}^{\infty} nP_n$ (P_n : probability that n customers are in the system)

Birth-and-Death Queueing Systems

1. m parallel, identical servers.
2. Infinite queue capacity (for now).
3. Whenever n users are in system (in queue plus in service) arrivals are Poisson at rate of λ_n per unit of time.
4. Whenever n users are in system, service completions are Poisson at rate of μ_n per unit of time.
5. FCFS discipline (for now).

The Fundamental Relationship



$P_n(t)$ = Prob [n users in system at time t]

$$P_n(t+\Delta t) = P_{n+1}(t)\cdot\mu_{n+1}\cdot\Delta t + P_{n-1}(t)\cdot\lambda_{n-1}\cdot\Delta t + P_n(t)\cdot[1-(\mu_n+\lambda_n)\cdot\Delta t]$$

The differential equations that determine the state probabilities

$$P_n(t+\Delta t) = P_{n+1}(t)\cdot\mu_{n+1}\cdot\Delta t + P_{n-1}(t)\cdot\lambda_{n-1}\cdot\Delta t + P_n(t)\cdot[1-(\mu_n+\lambda_n)\cdot\Delta t]$$

After a simple manipulation:

$$\frac{dP_n(t)}{dt} = -(\lambda_n+\mu_n)\cdot P_n(t) + \lambda_{n-1}\cdot P_{n-1}(t) + \mu_{n+1}\cdot P_{n+1}(t) \qquad (1)$$

(1) applies when n = 1, 2, 3,….; when n = 0, we have:

$$\frac{dP_0(t)}{dt} = -\lambda_0\cdot P_0(t) + \mu_1\cdot P_1(t) \qquad (2)$$

- The system of equations (1) and (2) is known as the Chapman-Kolmogorov equations for a birth-and-death system

The "state balance" equations

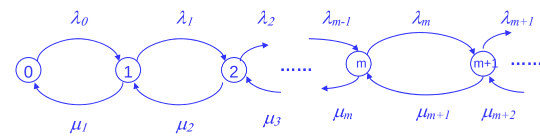
- We now consider the situation in which the queueing system has reached "steady state", i.e., t is large enough to have $P_n(t) = P_n$, independent of t, or $\frac{dP_n(t)}{dt} = 0$
- Then, (1) and (2) provide the state balance equations:

$$\lambda_0 \cdot P_0 = \mu_1 \cdot P_1 \qquad n = 0 \qquad (3)$$

$$(\lambda_n + \mu_n) \cdot P_n = \lambda_{n-1} \cdot P_{n-1} + \mu_{n+1} \cdot P_{n+1} \qquad n = 1, 2, 3,.. \qquad (4)$$

- The state balance equations can also be written directly from the state transition diagram

Birth-and-Death System: State Transition Diagram



- We are interested in the characteristics of the system under equilibrium conditions ("steady state"), i.e., when the state probabilities $P_n(t)$ are independent of t for large values of t
- Can write system balance equations and obtain closed form expressions for P_n, L, W, L_q, W_q

Solving…..

Solving (3) and (4), we have:

$$P_1 = \frac{\lambda_0}{\mu_1} \cdot P_0; \quad P_2 = \frac{\lambda_1}{\mu_2} \cdot P_1 = \frac{\lambda_1 \cdot \lambda_0}{\mu_2 \cdot \mu_1} \cdot P_0 \qquad etc.$$

and, in general,

$$P_n = \frac{\lambda_{n-1} \cdot \lambda_{n-2} \cdot ..... \cdot \lambda_1 \cdot \lambda_0}{\mu_n \cdot \mu_{n-1} \cdot ..... \cdot \mu_2 \cdot \mu_1} \cdot P_0 = K_n \cdot P_0$$

But, we also have: $1 = \sum_{n=0}^{\infty} P_n = P_0 \cdot (1 + \sum_{n=1}^{\infty} K_n)$

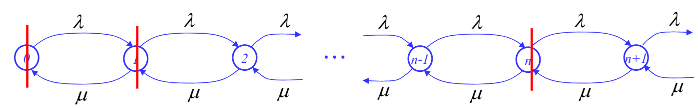
Giving, $P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} K_n}$

Condition for steady state: $\sum_{n=1}^{\infty} K_n < \infty$

M/M/1: Observing State Transition Diagram from Two Points

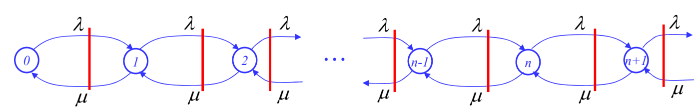
- From point 1:

$\lambda P_0 = \mu P_1 \quad (\lambda + \mu) P_1 = \lambda P_0 + \mu P_2 \qquad (\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$



- From point 2:

$\lambda P_0 = \mu P_1 \quad \lambda P_1 = \mu P_2 \qquad \lambda P_n = \mu P_{n+1}$



M/M/1: Derivation of P_0 and P_n

Step 1: $P_1 = \frac{\lambda}{\mu} P_0, \quad P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0, \; \cdots, \quad P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$

Step 2: $\sum_{n=0}^{\infty} P_n = 1, \;\Rightarrow\; P_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = 1 \;\Rightarrow\; P_0 = \frac{1}{\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n}$

Step 3: $\rho = \frac{\lambda}{\mu}$, then $\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = \sum_{n=0}^{\infty} \rho^n = \frac{1-\rho^{\infty}}{1-\rho} = \frac{1}{1-\rho} \quad (\because \rho < 1)$

Step 4: $P_0 = \frac{1}{\sum_{n=0}^{\infty} \rho^n} = 1-\rho$ and $P_n = \rho^n (1-\rho)$

M/M/1: Derivation of L, W, W_q, and L_q

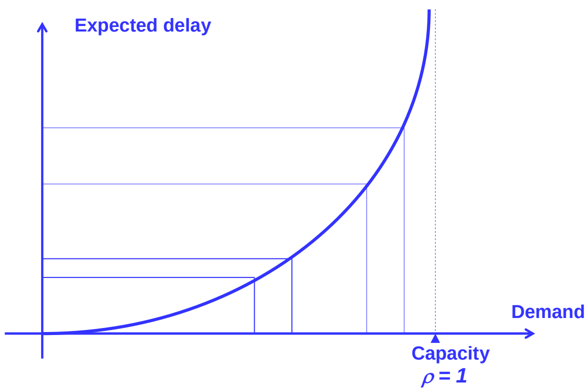
- $L = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n\rho^n (1-\rho) = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = (1-\rho)\rho \sum_{n=1}^{\infty} n\rho^{n-1}$

$$= (1-\rho)\rho \frac{d}{d\rho}\left(\sum_{n=0}^{\infty} \rho^n\right) = (1-\rho)\rho \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$= (1-\rho)\rho\left(\frac{1}{(1-\rho)^2}\right) = \frac{\rho}{(1-\rho)} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$

- $W = \frac{L}{\lambda} = \frac{\lambda}{\mu-\lambda} \cdot \frac{1}{\lambda} = \frac{1}{\mu-\lambda}$
- $W_q = W - \frac{1}{\mu} = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}$
- $L_q = \lambda W_q = \lambda \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\lambda^2}{\mu(\mu-\lambda)}$

High Sensitivity of Delay at High Levels of Utilization



M/M/1: An alternative, direct derivation of L and W

- **For an M/M/1 system, with FCFS discipline:**

$$W = \sum_{n=0}^{\infty} \frac{(n+1)}{\mu} \cdot P_n = E[\frac{N+1}{\mu}] = \frac{E[N]+1}{\mu} = \frac{L+1}{\mu} \qquad \textbf{(1)}$$

- **But from Little's theorem we also have:**

$$L = \lambda \cdot W \qquad \textbf{(2)}$$

- **It follows from (1) and (2) that, as before:**

$$L = \frac{\lambda}{\mu-\lambda}; \qquad W = \frac{1}{\mu-\lambda}$$

Does the queueing discipline matter?

Additional important M/M/1 results

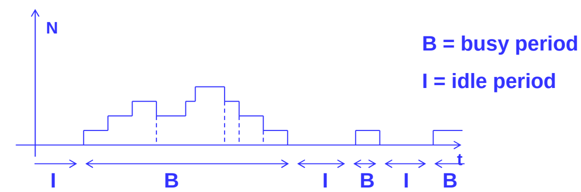
- **The pdf for the total time in the system, *w*, can be computed for a M/M/1 system (and FCFS):**

$$f_W(w) = (1-\rho)\mu e^{-(1-\rho)\mu w} = (\mu-\lambda)e^{-(\mu-\lambda)w} \text{ for } w \geq 0$$

Thus, as already shown, $W = 1/(\mu - \lambda) = 1/[\mu(1-\rho)]$

- **The standard deviation of N, w, N_q, w_q are all proportional to $1/(1-\rho)$, just like their expected values (L, W, L_q, W_q, respectively)**
- **The expected length of the "busy period", *E[B]*, is equal to $1/(\mu - \lambda)$**

M/M/1: *E*[*B*], the expected length of a busy period



$$P_0 = \frac{E[\text{length Idle period}]}{E[\text{length Busy period}] + E[\text{length Idle period}]}$$

But, $P_0 = 1-\rho \qquad E[\text{length Idle period}] = 1/\lambda$

Therefore, $E[B] = E[\text{length Busy period}] = \frac{1}{\mu}\cdot\frac{1}{(1-\rho)} = \frac{1}{\mu-\lambda}$