Quiz # 2
In-class, open books and notes

Problem 1 (40 Points)

An environmental variable X has value X_i in day i. Due to budgetary constraints, X is measured only every third day. If X is measured in day i, then the observed value of X_i is used to estimate the three-day average $\overline{X} = \frac{1}{3}(X_{i-1} + X_i + X_{i+1})$. Given that the variables X_i have multivariate normal distribution with common mean value m and common variance σ^2 and that the correlation coefficient between X_i and X_j is $\rho_{ij} = 0.9^{|i-j|}$,

(a) Find the distribution of $(\overline{X} \mid X_i)$. [Hint: start by finding the joint distribution of $\overline{X}$ and X_i].

(b) Based on the answer to (a), suggest a better estimator of $\overline{X}$ from X_i.

Problem 2 (30 Points)

In a river reach, flooding occurs if the water level H exceeds 8 meters. Following a heavy rainstorm, the water level is $H = H_o + 0.5\, I\, D^{0.8}$, where H_o is the water level before the storm, I is storm depth in meters/hour (integrated over the river catchment) and D is the storm duration in hours. Suppose that H_o is known and equal to 3 meters and that ln (I) and ln (D) are independent normal variables with distributions

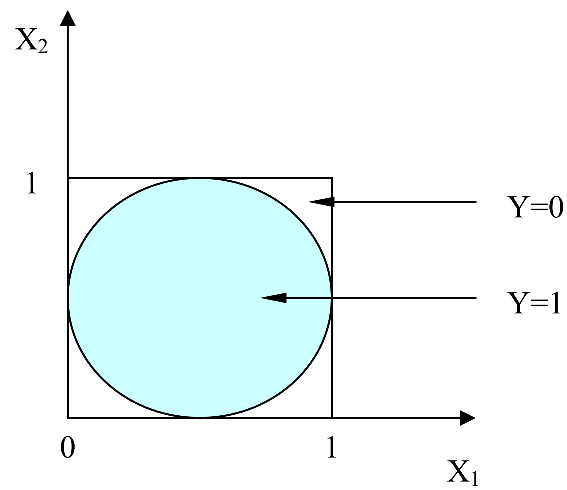
$\ln(I) \sim N(0, 1)$
$\ln(D) \sim N(1, 1)$

where $N(m, \sigma^2)$ is the normal distribution with mean value m and variance σ^2

(a) Find the distribution of $\ln(I\, D^{0.8})$.
(b) State the condition for flooding in terms of ln (I) and ln (D), and evaluate the second moment reliability index β in the [ln (I), ln (D)] plane. For this problem, is Φ(-β) the exact probability of flooding?

Problem 3 (30 Points)

Using a computer, one can simulate random points with uniform distribution inside the unit square $\{0 \le X_1 \le 1, 0 \le X_2 \le 1\}$.



Let Y be an indicator variable as follows:

Y = 1 if the point is inside the inscribed circle (see the figure above)
Y = 0 otherwise

(a) Show that $E[Y] = \frac{\pi}{4}$

One may estimate $E[Y]$ (hence $\frac{\pi}{4}$) as the sample average:

$$\overline{Y} = \frac{1}{n}\sum_i Y_i$$

where $\{Y_1, \ldots\ldots\ldots, Y_n\}$ is a random sample from the distribution of Y obtained through repeated computer simulations.

(b) Find the mean value and variance of $\overline{Y}$

(c) What sample size n is needed to estimate $\frac{\pi}{4}$ with a standard deviation of the estimation error equal to 0.001?