1.061 / 1.61 Transport Processes in the Environment
Fall 2008

Answer 5.1.

First, determine if the flow is laminar. The velocity in the tube is U = Q/A = 0.1 cm/s. The diameter of the tube is $d = \sqrt{(4/\pi)A} = 1.13$ cm. The Reynolds number is Re = UD/ν = $(0.1\ \text{cm/s})(1.13\ \text{cm})/(0.01\ \text{cm}^2\text{s}^{-1}) = 11.3 << 2100$, the limit for laminar flow.

Here, Pe = UL/D = $(0.1\ \text{cm/s})(100\ \text{cm})/(10^{-5}\ \text{cm}^2\text{s}^{-1}) = 10^6$. Since Pe >>1 the arrival of the dye at L can be estimated by L/U = 1000 s. At this time the dye patch will have a length, $4\sigma = 4\sqrt{2Dt} = 4\sqrt{2(10^{-5}\text{cm}^2\text{s}^{-1})(1000\text{s})} = 0.57$ cm. The maximum concentration will occur when the center of mass passes that point, i.e. at t = L/U. So, the maximum concentration will be C (x = L, t = L/U = 1000s). Since the mass mixes instantly across the cross-section, and the flow is uniform across the section, the system is effectively one dimensional. The concentration resulting from an instantaneous point release is

$$C(x,t) = \frac{M}{A_{yz}\sqrt{4\pi Dt}} \exp(-(x-ut)^2/4Dt).$$

Evaluated at x = L and t = L/U, we have C (x =L, t =L/U) =

$$\frac{M}{A\sqrt{4\pi DL/U}} = \frac{1\ \text{g}}{1\text{cm}^2\sqrt{4\pi(10^{-5}\text{cm}^2\text{s}^{-1})(100\ \text{cm})/(0.1\ \text{cm/s})}} = 28.2\ \text{g cm}^{-3}$$

Answer 5.2.

a) and b) For all cases the length scale is L = 100 m and the diffusion rate $D = 1\text{m}^2\text{s}^{-1}$.

U [m s-1]	Diffusion: $T_D = L^2/8D$	Advection: T_U=L/U	Pe = UL/D	Curve
0.001	1250 s	100,000 s	0.1	Blue
0.1	1250 s	1,000 s	10	Green
1	1250 s	100 s	100	Red

c) For Fickian diffusion, the peak concentration occurs at the center of mass which arrives at the advection time scale, i.e. at time T_U the peak concentration in the cloud will be located at L. However, the peak concentration observed at L will not necessarily occur at T_U. This is because the magnitude of the concentration is changing as the cloud passes the measurement point. If the cloud is passing very slowly (Pe << 1), the concentration may decline considerably as the cloud is passing, such that the peak concentration seen at L will occur before the center of mass arrives. If the concentration changes very little as the cloud passes (advection is faster than diffusion, Pe >>1) the peak concentration observed at L will correspond to the advection time scale. We expect this to occur for the Red and possibly the Green systems.

To find the concentration we must first determine whether the cloud is three-, two-, or one-dimensional as it reaches the measurement position. This will depend on the time required to mix the dye across the channel area, $A = 10\ m^2$. If we assume a square cross-section, then the width and depth of the channel is, $L_y = L_z = \sqrt{10}$ m. The mixing time-scale (see chap 4, eq 24), is $t_i = L_i^2 / 4D_i = 10m^2/(4\ m^2s^{-1}) = 2.5$ s. For each curve both T_D and $T_U >> 2.5$ s, so we can safely assume that the dye is well-mixed across the channel when it arrives at L. Thus, we can use a one-dimensional solution to estimate the concentration observed at L at t = L/U.

Red Curve: $C\,(\,x = L,\ t = L/U\,) = \dfrac{M}{A\sqrt{4\pi Dt}} = \dfrac{1000g}{10m^2\sqrt{4\pi(1m^2s^{-1})(100s)}} = 2.82\ gm^{-3}$.

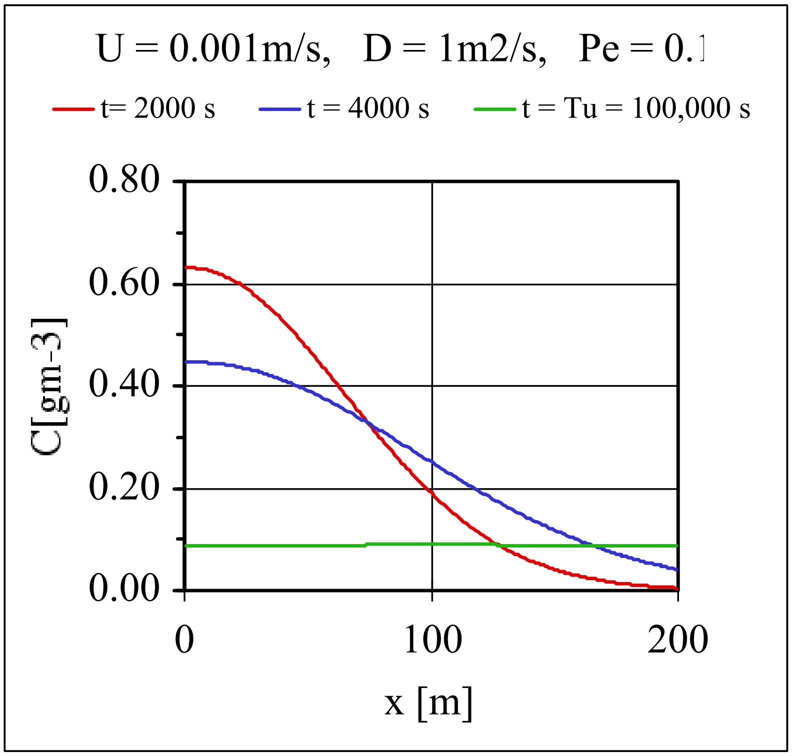
This value is consistent with the peak concentration observed at L.

For the Green Curve $C\,(\,x{=}L,\ t{=}L/U\,) = \dfrac{1000g}{10m^2\sqrt{4\pi(1m^2s^{-1})(1000s)}} = 0.89\ gm^{-3}$

This value is also consistent with the peak observed concentration.

For the Blue Curve $C\,(\,x{=}L,\ t{=}L/U\,) = \dfrac{1000g}{10m^2\sqrt{4\pi(1m^2s^{-1})(100{,}000s)}} = 0.09\ gm^{-3}$

This is LESS than the concentration observed at L at t = 4000s ($C = 0.25\ gm^{-3}$). For this case the peak concentration observed at L occurs before T_U, i.e. before the center of mass passes L. This is because the concentration in the cloud is dropping off faster (via diffusion) than the center of mass can travel (via advection), consistent with Pe << 1. The spatial distribution, C(x), for this condition is shown below.



d) We are interested in the concentration observed at L = 100m. In all cases the transport time scale for this location is 100 s or more, which is much longer than the injection time scale. For this reason the injection can be assumed to be instantaneous.

Answer 5.3

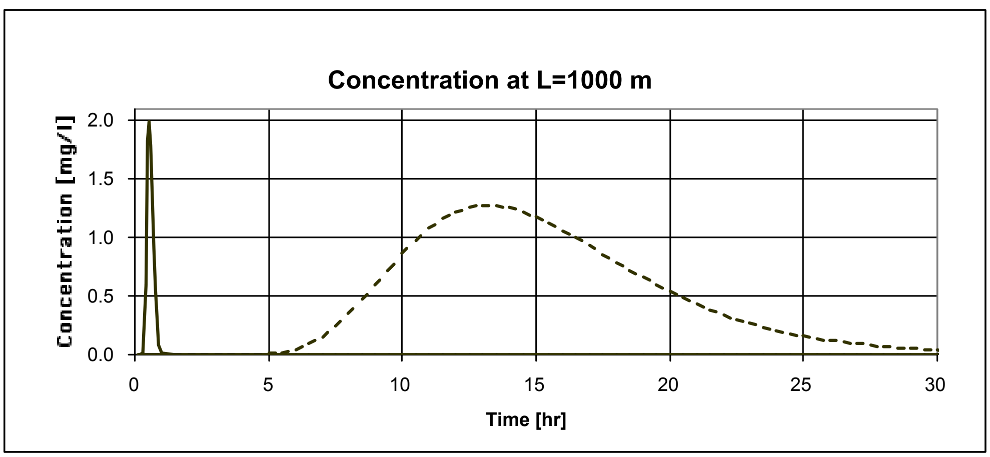
Start by defining the system dynamics using time scales
We are interested in predicting the concentration at L =1000 m. Because 1) the channel is square; 2) $D_y = D_z$, and 3) the discharge is at the center of the channel, the time to mix across the cross-section is, $T_{mix} = 20m^2/(4D_y)$. All time scales are given in seconds.

Case	U $[m\ s^{-1}]$	D_y $[m^2 s^{-1}]$	Mixing $T_{mix}=20m^2/4D_y$	D_x $[m^2 s^{-1}]$	Diffusion $T_D=L^2/8D_x$	Advection $T_U=L/U$	$Pe=LU/D_x$
1	0.5	0.1	50 s	10	12500 s	2000 s	50
2	0.02	0.01	500 s	1	125000 s	50,000 s	20

In both cases $T_{mix} << T_D$ and T_U, so we can assume that the chemical is uniformly distributed in y and z long before it reaches the hatchery. Thus, we can use a 1-D solution to describe the concentration experienced at L = 1000 m. In addition, the release time [100 s] is also very short compared to either transport time scale, which justifies the use of an instantaneous release model. Finally, since concentration is uniform in y and z, no images are needed to correct for the no-flux boundaries, i.e. the condition $\partial C/\partial y = 0$ and $\partial C/dz = 0$ is met at the boundaries because it is true everywhere after $t = T_{mix}$. With these approximations we represent the concentration at the hatchery as

$$C(x = L, t) = \frac{M}{A\sqrt{4\pi D_x t}} EXP\left(-\frac{(L - ut)^2}{4 D_x t}\right).$$

The total mass released is $M = (2m^3s^{-1})(0.1g\ l^{-1})(100\ s)(1000\ l\ m^{-3}) = 20000$ g or 20 kg. The solution C (x = L, t) is plotted below for case 1 [solid line] and case 2 [dashed line]



From the above graph, we describe the following exposure conditions. For Case 1, the high flow conditions, the exposure concentration has a peak of 2 mgl^{-1} and it exceeds the standard for about 1/2 hour. For Case 2, the low flow conditions, the exposure concentration has a peak of 1.3 mgl^{-1} and it exceeds the standard for 13 hours. In both cases the exposure is not chronic exposure therefore the application of the standard is ambiguous. However, if it is feasible to shut off the hatchery intake for the duration of exposure, this is the safest action to take.

We could also estimate the peak and duration of exposure as follows.
Since in both cases Pe >1, the peak arrives at the advection time scale, i.e.

$$C_{max} = C(x=L, t = T_u) = \frac{M}{A\sqrt{4\,\pi\,D_x T_U}}$$

$$\text{For Case 1: } C_{max} = \frac{20,000\text{ g}}{20\text{m}^2\sqrt{4\,\pi\,(10\text{ m}^2\text{s}^{-1})(2000\text{s})}} = 2\text{ gm}^{-3} = 2\text{ mgl}^{-1}$$

$$\text{For Case 2: } C_{max} = \frac{20,000\text{ g}}{20\text{m}^2\sqrt{4\,\pi\,(1\text{ m}^2\text{s}^{-1})(50,000\text{s})}} = 1.3\text{ gm}^{-3} = 1.3\text{ mgl}^{-1}$$

At the time the peak passes, the chemical cloud has length, $4\sigma = 4\sqrt{2D_x T_U}$. If we assume that the cloud does not grow as it passes x = L, then this translates into a duration of $\Delta T = 4\sigma/U$.

Case 1: $\Delta T = 4\sigma/u = \left(4\sqrt{2D_x T_U}\right)/u = \left(4\sqrt{2(10\text{m}^2\text{s}^{-1})2000\text{s}}\right)/(0.5\text{m/s}) = 1600\text{ s} = 27\text{ min.}$
Case 2: $\Delta T = \left(4\sqrt{2D_x T_U}\right)/u = \left(4\sqrt{2(1\text{m}^2\text{s}^{-1})50,000\text{s}}\right)/(0.02\text{m/s}) = 63,246\text{ s} = 17.6\text{ hours.}$

Answer 5.4

a) Since the system is unbounded in each coordinate direction (x, y, z), the concentration can never become uniform in any direction and a three-dimensional solution will apply. Assuming the release occurs at t = 0 s, the trajectory of the chemical cloud's center of mass is (x = Ut, y = H, z = 0). To satisfy the no-flux boundary at y = 0, we must add an image source at y = -H. The center of mass of the image source follows the trajectory (x = Ut, y = -H, z = 0). With these conditions, and isotropic diffusivity the concentration solution is

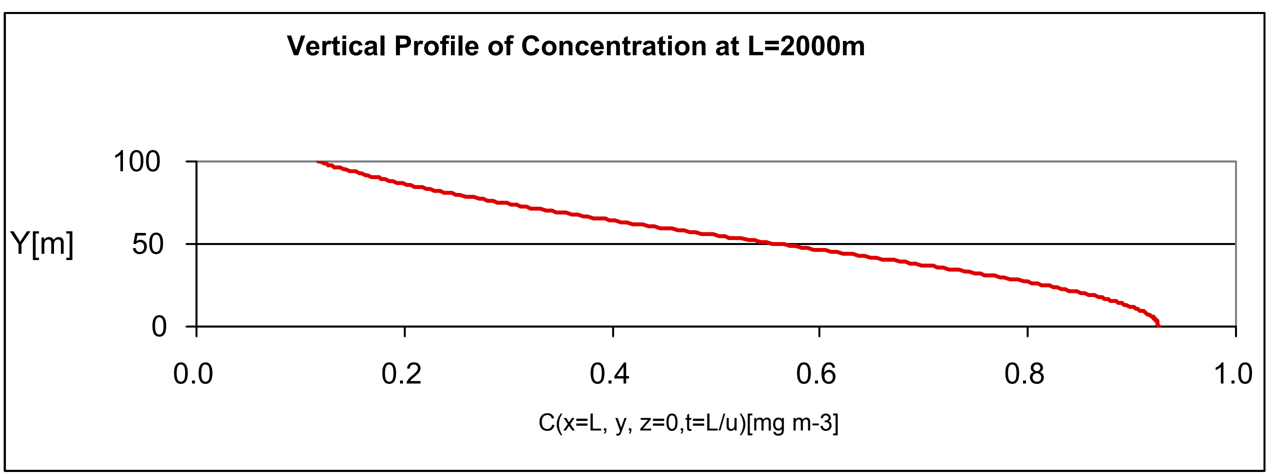
C (x, y, z, t) =

$$\frac{M}{(4\pi Dt)^{3/2}}\left[\exp\left(-\frac{(x-ut)^2+(y-H)^2+z^2}{4Dt}\right)+\exp\left(-\frac{(x-ut)^2+(y+H)^2+z^2}{4Dt}\right)\right].$$

b) Given the length scale x = L = 500 m, Pe# = (2 ms^{-1})(500m)/(1m^2s^{-1}) = 1000 >> 1. This tells us that transport from the smokestack to this position (x = 500 m) is dominated by advection.

c) The release will appear to be instantaneous if the time scale for release, T_R, is much shorter than the time scale of transport. At distances for which Pe# = Ux/D >> 1, the transport is dominated by advection. This is true for x > > D/u = 0.5 m, essentially the entire flow domain. Then, the release will appear to be instantaneous at distances for which $T_U >> T_R$. Or, $x/U >> T_R$. This is true for x >> (300 s)(2 ms^{-1}) = 600 m.

d) At x = 2000 m the peak concentrations should arrive at T_u = 2000m/(2ms^{-1})=1000 s. The vertical profile at x = L, t = L/u is shown below for 2000m. Note that the peak concentration occurs at the no-flux boundary and not at the height of the release.



Answer 5.5.

a. Assume the flow fills the channel uniformly, then U =Q/A = 2.5 x 10^{-4} ms^{-1}. Using L = 75, Pe = 0.19. Alternatively, for L = (75-25) = 50 m, Pe = 0.13. In either case the Peclet number indicates that the system is dominated by diffusion.

b. According to the Peclet number, transport is dictated by diffusion. The time-scale for the contaminant to reach the harbor can be estimated as the time-scale for diffusive transport over the 50-m between the spill and the harbor entrance. $T_D = (50m)^2/(8 \times 0.1\ m^2s^{-1}) = 3125\ s \approx 1hr$. So, I have one hour to put up a contaminant-absorbing barrier and protect the harbor

c. In the vertical, $t_{mix}=(2\ m)^2/(4 \times 0.1\ m^2s^{-1}) = 10\ s$
In the lateral, $t_{mix}=(10\ m)^2/(4 \times 0.1\ m^2s^{-1}) = 250\ s$

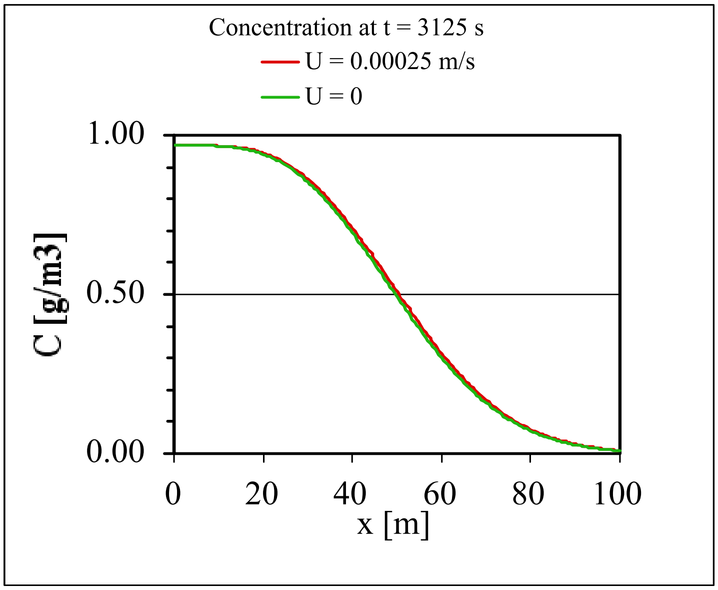
d. Since the mixing time scales in both the lateral and vertical are much shorter than the time required for the contaminant to reach the harbor, we can assume the contaminant is mixed across the channel area when it reaches x = 75 m. If concentration is uniform (well-mixed) in the lateral and vertical, we can drop these two dimensions, and use a one-dimensional solution.

e.
$$C(x,t) = \frac{M}{A_{yz}\sqrt{4\pi Dt}}\left[\underbrace{\exp\left(-\frac{(x-25\ -\ Ut)^2}{4Dt}\right)}_{\text{real source}} + \underbrace{\exp\left(-\frac{(x+25\ -\ Ut)^2}{4Dt}\right)}_{\text{image source}}\right]$$

The image source is needed to satisfy the no-flux boundary at x = 0. Since Pe<<1, we could also neglect U entirely and still get a good representation of C(x,t).

$$C(x,t) = \frac{M}{A_{yz}\sqrt{4\pi Dt}}\left[\exp\left(-\frac{(x-25)^2}{4Dt}\right) + \exp\left(-\frac{(x+25)^2}{4Dt}\right)\right]$$

A comparison of the full solution (U = 0.00025 m/s) and the solution that neglects U is given below. One can quickly see that U is indeed negligible, as implied by Pe.



f. U and D are assumed to be uniform, i.e. not functions of (x, y, z), but in fact the no-slip condition at the channel boundaries will make U = f(y,z).
We assume that D is isotropic. In fact, turbulent diffusion in the longtudinal direction will be much more rapid than in the vertical or horizontal. This is discussed further in Chapter 9. We neglect losses to the atmosphere, when in fact the gasoline is volatile. We assume that no gasoline absorbs to the sediment. We assume that the flow and thus velocity are not functions of time.