Reflections: Design Exercise 2
1.050 Solid Mechanics
Fall 2004

Reflections on Design Exercise #2, a truss design with cost considerations

We consider a series of simple trusses, all spanning the length L, (L= 20ft.) with diagonals inclined at some angle theta. We allow the number of bays to vary from *n=1* to some finite but possibly large number. As the number of bays, or sections, increases, the height of the truss, *h,* must decrease in accord with the following:

Since $h = \frac{a}{2} \cdot \tan\theta$ and $L = a \cdot n$ we have that

$$h = \frac{L}{2n} \cdot \tan\theta$$

The figure at the right shows the first four trusses in the sequence.

Cutting a section to determine the forces in the critical members - those at mid span - and applying force and moment equilibrium yields the following expressions for the compressive force in the top member, f_t, the force in the diagonals, f_d, and the tensile force in the bottom member, f_b, for a truss with *n* bays, **n even**[1]

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$$f_t = \frac{n \cdot W}{2\tan\theta}$$

$$f_d = \frac{W}{2\sin\theta}$$

$$f_b = \frac{(n-1) \cdot W}{2\tan\theta}$$

For **_n_ odd**, the two top members aside the load show the same force as f_b above (but in compression), while the central member below the loaded node shows the same force as f_t above (but in tension). So the max. magnitude, axial load in the horizontal members is
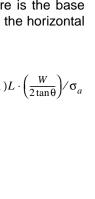
$$|f|_{max} = \frac{n \cdot W}{2\tan\theta}$$

while the diagonal members experience the same load magnitude, regardless the number of bays.

We consider the cost of the structure, taken to be proportional to the total volume of materials. The total length of the horizontal members is *(2n-1)a* where a, shown in the figure is the base length of one of the bays. This holds for *n* even or odd. The cross-sectional area of the horizontal elements is taken to be just that which will not fail at the allowable stress - σ_a.

$$A_h = F_{max}/\sigma_a \quad \text{or} \quad A_h = \left(\frac{n \cdot W}{2\tan\theta}\right)/\sigma_a$$

So the volume of the horizontal members is $V_h = (2n-1)a \cdot \left(\frac{n \cdot W}{2\tan\theta}\right)/\sigma_a = (2n-1)L \cdot \left(\frac{W}{2\tan\theta}\right)/\sigma_a$ where we have used the fact that *L = a*n*.

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1. We show an isolation for the case *n= 4.*

Likewise, the number of diagonal members is just *2n, n* odd or even. And again, we take the area to be that which will produce a stress equal to the allowable when the structure is fully loaded.

So $A_d = F_{max}/\sigma_a$ or $A_d = (\frac{W}{2\sin\theta})/\sigma_a$. The volume of the diagonal members is then

$$V_d = 2n\left(\frac{a}{2\cos\theta}\right)\cdot\left(\frac{W}{2\sin\theta}\right)/\sigma_a = \frac{L}{\cos\theta}\cdot\left(\frac{W}{2\sin\theta}\right)/\sigma_a$$ where we have again used the fact that *L = a*n*

after determining the length of the diagonal in terms of *a, i.e.,* $l_d = \left(\frac{a}{2\cos\theta}\right)$.

So the total volume is

$$V_{total} = \frac{L}{\cos\theta}\cdot\left(\frac{W}{2\sin\theta}\right)/\sigma_a + (2n-1)L\cdot\left(\frac{W}{2\tan\theta}\right)/\sigma_a$$

or

$$V_{total} = \left(\frac{WL}{2\sigma_a}\right)\frac{1}{\sin\theta}\cdot\left[\frac{1}{\cos\theta} + (2n-1)\cos\theta\right]$$

With cost proportional to total volume, cost is a function of *n* and θ. For any theta, we would set *n=1* to achieve a minimum. This, for the exercise as laid out in class, was left as a possibility since we set no explicit limit on height of the truss. More than likely, there would be a limit on overhead room and we would be constrained to take *n* greater than 1. Note that the height of the truss is given by $h = \frac{a}{2}\cdot\tan\theta = \frac{L}{2n}\cdot\tan\theta$.

Regardless of the number of bays, we can seek the value(s) of θ which minimizes the following function:

$$y(\theta) = \frac{1}{\sin\theta}\cdot\left[\frac{1}{\cos\theta} + (2n-1)\cos\theta\right]$$

Setting *dy/d*θ =0 yields root(s) for θ for minimum y when: $\boxed{(\tan\theta)^2 = 2n}$

The table below shows optimum q for any given n, the number of bays

n	θ	*h/L*
1	54.7º	0.707
2	63.4º	0.500
3	67.8º	0.408
...	...	
8	75.9º	0.250

Many of you, using a spread sheet, sought and found these same values. One of the purposes of this note is to show how one can obtain the conditions for minimum cost using the calculus.

Another purpose was to illustrate the relative orders of magnitude of the forces in the diagonal members and the forces in the top and bottom horizontal members. We see that as the number of bays increases, (the height of the truss decreases), the forces in the top and bottom members increase accordingly (proportional to *n*) while the force in the diagonal does not change - as long as we do not change the loading geometry and magnitude. We say the diagonals "pick up the shear force", which for this loading condition does not vary over the span if we imagine the truss to be a beam.