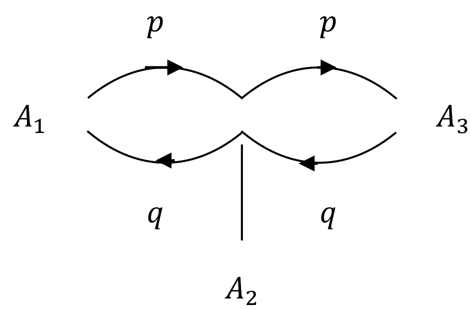
Problem set #1 for 5.72 (spring 2012) Cao

I. Consider a random walk on a linear three-site model. p and q are the probabilities of moving right and left respectively. $(p+q=1)$



1) Write the transition matrix Q.
2) Find the stationary distribution and show that it satisfies detailed balance.
3) For the special case of $p=q=\frac{1}{2}$, compute the probability at $n=3$ after s steps, $p_n(s)$, given $p_n(0)=\delta_{n1}$.
4) *Repeat the calculation in 3) for $p \neq q$.

II. A one-dimensional random walk on an infinite lattice is described by a master equation

$$\frac{dp_n}{dt} = a(p_{n-1}-p_n) + b(p_{n+1}-p_n),$$

where a and b are the forward and backward rate constants.

1) Calculate:

$$\langle n(t)\rangle = \sum_n n p_n(t)\,,$$

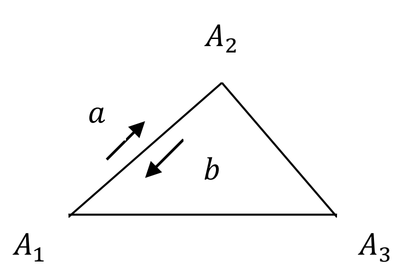
$$\langle n^2(t)\rangle = \sum_n n^2 p_n(t)\,,$$

and

$$\langle \delta n^2 \rangle = \langle n^2(t)\rangle - \langle n(t)\rangle^2.$$

2) *Given that the random walker starts at $n=0$, i.e. $p_n(0)=\delta_{n0}$, find $p_n(t)$.

III. For the three-site model the forward rate is a and the backward rate is b. The walker is initially at site A_1.



1) Calculate the probability that the walker has not arrived at A_3 by time t.
2) Calculate the mean first passage time from A_1 to A_3.

IV. *Consider a random walk on an infinite d-dimensional lattice with transition rate to the next neighbor being α. The random walker starts at $n = 0$, i.e. $p_n(0) = \delta_{n0}$.

1) Derive a formal expression for $p_n(t)$.
2) Show the long-time behavior is diffusive with diffusion constant α and $\lim_{t\to\infty} p_0(t) \propto t^{-\frac{d}{2}}$.
3) $f_0(t)$ is the probability that the walker arrives at site 0 for the first time. Show that

$$\lim_{s\to 0} \hat{f}_0(s) = 1 - \frac{1}{\alpha \hat{p}_0(s)},$$

where $\hat{f}_0$ and $\hat{p}_0$ are the Laplace transforms of f_0 and p_0.

4) Define the total return probability as $R = \int_0^\infty f_0(t)dt$.

Show that $R = 1$ for $d = 1$ or 2

and $R < 1$ for $d > 2$.

(Note: references Reichl and Montroll.)

* honorary problems will not be counted towards grade.

5.72 Statistical Mechanics
Spring 2012