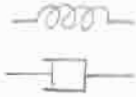
BEH. 410 - 03/19/2003

L# 13

Last time: Strain energy density function

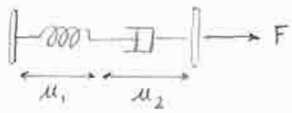
Linear viscoelasticity
- creep
- stress relaxation
- oscillatory testing

Lumped parameter models:
- Maxwell



$F_1 = k\, u_1$ spring → elastic

$F_2 = \eta \dfrac{du_2}{dt}$ dashpot → viscous



□ Lumped parameter models

Maxwell

$$\left.\begin{aligned} u &= u_1 + u_2 \\ F &= F_1 = F_2 \end{aligned}\right\} \quad \boxed{k \frac{du}{dt} = \frac{dF}{dt} + \frac{k}{\eta} F}$$

- stress relaxation

real tissue not very well described by Maxwell model

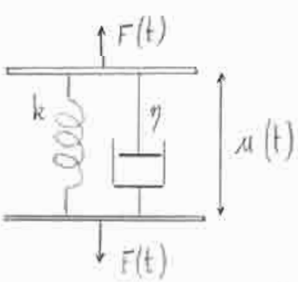
- limitations of the Maxwell model:
  - stress relaxes to zero
  - creep not like most material

- creep

note: F related to stress σ & displacement u related to strain ε

Try another model with paradigm:
- choose model that fits experimental behavior
- determine $k_1, \dots, \eta_1, \dots$
- relate to microscopic behavior

Voigt

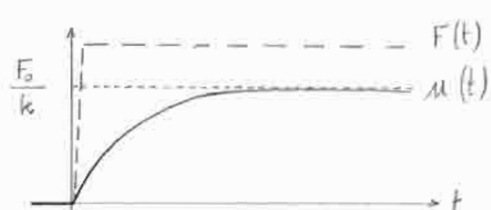


$$\left.\begin{aligned} u &= u_1 = u_2 \\ F &= F_1 + F_2 \\ &= k\, u + \eta \frac{du}{dt} \end{aligned}\right\} \quad \boxed{\frac{F}{\eta} = \frac{k}{\eta} u + \frac{du}{dt}}$$

- creep: given forcing step function for F, find $u(t)$

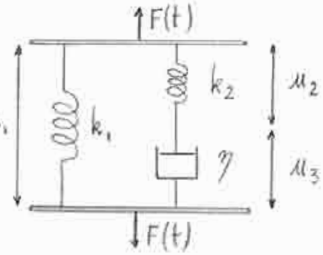
Solution to the constitutive equation:

$$\boxed{\begin{aligned} u(t) &= \frac{F_o}{k}\left(1 - \exp\left(\frac{-t}{T_R}\right)\right) \\ T_R &= \eta / k \end{aligned}}$$



no initial deflection

Kelvin or "standard linear solid



$$\left.\begin{array}{l} u = u_1 = u_2 + u_3 \\ F = F_1 + F_2 \\ F_2 = F_3 \end{array}\right\} \quad \boxed{F + \alpha \frac{dF}{dt} = k_1 u + \beta \frac{du}{dt}}$$

In pb set # 5, find α & β in terms of k_1, k_2 & η

- The standard linear solid exhibits creep & stress relaxation similar to most biological tissues

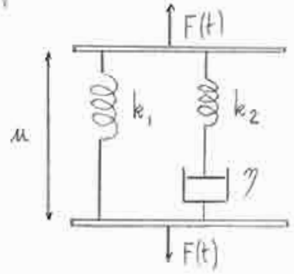
Continue to add elements . $k_1, k_2, \dots, k_n, \eta_1, \eta_2, \dots \eta_m$

each one corresponds to a characteristic relaxation time

you can fit about anything, but what do you learn about your biological tissue?

- Dynamic oscillatory behavior (transient behavior, sinusoidal steady - state)

▷ example of the 3- element model



input $u(t) = u_o \cos \omega t = \mathcal{R}e\left[u_o e^{i\omega t}\right]$

$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$

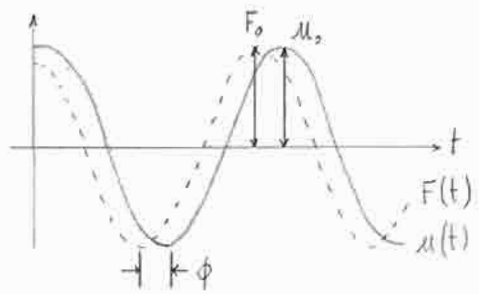
output $F(t) = \mathcal{R}e\left[\hat{F}(\omega) e^{i\omega t}\right] = \mathcal{R}e\left[\underbrace{|\hat{F}|}_{F_o} \underbrace{e^{i\phi} e^{i\omega t}}_{\exp(i(\omega t + \phi))}\right]$

$\hat{F}(\omega)$ ↓ function of ω, k_1, k_2, η

$F(t) = F_o \cos(\omega t + \phi)$

define a complex modulus

$$\hat{G}(\omega, k_1, k_2, \eta) = \frac{\hat{F}(\omega)}{u_o} = \frac{|\hat{F}| e^{i\phi}}{u_o} = \frac{F_o(\omega)}{u_o}(\cos\phi + i \sin\phi)$$



$$\left\{\begin{array}{l} G' \text{ storage modulus } \equiv \dfrac{F_o(\omega)}{u_o} \cos\phi \\ G'' \text{ loss modulus } \equiv \dfrac{F_o(\omega)}{u_o} \sin\phi \\ \hat{G} = G' + i\, G'' \quad \text{and} \quad \dfrac{G''}{G'} = \tan\phi \end{array}\right.$$

- Experiment . measure frequency - dependence of G' and G" (or phase lag ϕ) and infer values for k_1, k_2, η , and finally relate them to molecular behavior

For SLS model $F + \alpha \dfrac{dF}{dt} = k_1 u + \beta \dfrac{du}{dt}$ and $\dfrac{d}{dt} \longleftrightarrow i\omega$

$$\hat{F}(\omega)(1 + i\alpha\omega)\cancel{e^{i\omega t}} = u_o (k_1 + i\beta\omega)\cancel{e^{i\omega t}}$$

we have $\hat{G}(\omega) = \dfrac{\hat{F}(\omega)}{u_o} = G' + i\, G''$

$$= \frac{k_1 + i\beta\omega}{1 + i\alpha\omega} = \frac{(k_1 + i\beta\omega)(1 - i\alpha\omega)}{(1 + i\alpha\omega)(1 - i\alpha\omega)} = \underbrace{\frac{k_1 + \alpha\beta\omega^2}{1 + \alpha^2\omega^2}}_{G'} + i \underbrace{\frac{(\beta - k_1\alpha)\omega}{1 + \alpha^2\omega^2}}_{G''}$$

{ low ω or long time
{ high ω or short time

at low frequency { dashpot has no effect because has plenty of time to respond
only k_1 counts , thus $G' = k_1$ (all force released in k_2 by η).
loss modulus governed by dashpot in the middle
very low dissipation when dashpot neglected

at high frequency { dashpot doesn't have time to respond , only the two springs count
no role for dashpot hence $G'' \rightarrow 0$
k_1 and k_2 act together , so $G' = k_1 + k_2$

the critical frequency where dashpot plays a role with be $\sim$ "$\frac{k}{\eta}$"