Numerical Methods for PDEs

Integral Equation Methods, Lecture 4
Formulating Boundary Integral Equations

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1 Outline

Slide 1

Laplace Problems
Exterior Radiation Condition
Green's function
Ansatz or Indirect Approach
Single and Double Layer Potentials
First and Second Kind Equations
Greens Theorem Approach
First and Second Kind Equations

2 3-D Laplace Problems

2.1 Differential Equation

Slide 2

Laplace's equation in 3-D

$$\nabla^2 u(\vec{x}) = \frac{\partial^2 u(\vec{x})}{\partial x^2} + \frac{\partial^2 u(\vec{x})}{\partial y^2} + \frac{\partial^2 u(\vec{x})}{\partial z^2} = 0$$

where

$$\vec{x} = x, y, z \in \Omega$$

and Ω is bounded by Γ.

2.2 Boundary Conditions

Slide 3

Dirichlet Condition

$$u(\vec{x}) = u_\Gamma(\vec{x}) \quad \vec{x} \in \Gamma$$

OR

Neumann Condition

$$\frac{\partial u(\vec{x})}{\partial n_{\vec{x}}} = \frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} \quad \vec{x} \in \Gamma$$

PLUS

A Radiation Condition

2.2.1 Radiation Condition

Slide 4

The Radiation Condition

$$lim_{\|\vec{x}\| \to \infty} u(\vec{x}) \to 0$$

not specific enough! Need

$$lim_{\|\vec{x}\| \to \infty} u(\vec{x}) \to O(\|\vec{x}\|^{-1})$$

OR

$$lim_{\|\vec{x}\| \to \infty} u(\vec{x}) \to O(\|\vec{x}\|^{-2})$$

2.3 Greens Function

SLIDE 5

Laplace's Equation Greens Function

$$\nabla^2 G(\vec{x}) = 4\pi\delta(\vec{x})$$

$\delta(\vec{x}) \equiv$ impulse in 3-D
Defined by its behavior in an integral

$$\int \delta(\vec{x}')f(\vec{x}')d\Omega' = f(0)$$

Not too hard to show

$$G(\vec{x}) = \frac{1}{\|\vec{x}\|}$$

3 Ansatz (Indirect) Formulations

3.1 Single Layer Potential

SLIDE 6

Consider

$$u(\vec{x}) = \int_{\Gamma} \frac{1}{\|\vec{x} - \vec{x}'\|}\sigma(\vec{x}')d\Gamma'$$

$u(\vec{x})$ automatically satisfies $\nabla^2 u = 0$ on Ω.

Must now enforce boundary conditions

3.1.1 Boundary Conditions

SLIDE 7

Dirichlet Problem

$$u_{\Gamma}(\vec{x}) = \int_{\Gamma} \frac{1}{\|\vec{x} - \vec{x}'\|}\sigma(\vec{x}')d\Gamma' \quad \vec{x} \in \Gamma$$

Neumann Problem

$$\frac{\partial u_{\Gamma}(\vec{x})}{\partial n_{\vec{x}}} = \frac{\partial}{\partial n_{\vec{x}}} \int_{\Gamma} \frac{1}{\|\vec{x} - \vec{x}'\|}\sigma(\vec{x}')d\Gamma' \quad \vec{x} \in \Gamma$$

3.1.2 Care Evaluating Integrals

SLIDE 8

On a smooth surface:

$$\lim_{x \to \Gamma} \frac{\partial}{\partial n_{\vec{x}}} \int_{\Gamma} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma'$$

$$= 2\pi\sigma(\vec{x}') + \int_{\Gamma} \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma'$$

3.1.3 Neumann Problem 2nd Kind!

SLIDE 9

$$\frac{\partial u_{\Gamma}(\vec{x})}{\partial n_{\vec{x}}} = 2\pi\sigma(\vec{x}') + \int_{\Gamma} \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma'$$

3.1.4 Radiation Condition

SLIDE 10

$$lim_{\|\vec{x}\| \to \infty} u(\vec{x}) = \int_{\Gamma} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma' \to O(\|\vec{x}\|^{-1})$$

Unless

$$\int_{\Gamma} \sigma(\vec{x}') d\Gamma' = 0$$

Then

$$lim_{\|\vec{x}\| \to \infty} u(\vec{x}) \to O(\|\vec{x}\|^{-2})$$

3.2 Double Layer Potential

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Consider

$$u(\vec{x}) = \int_{\Gamma} \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x} - \vec{x}'\|} \mu(\vec{x}') d\Gamma'$$

$u(\vec{x})$ automatically satisfies $\nabla^2 u = 0$ on Ω.

Must now enforce boundary conditions

3.2.1 Boundary Conditions

SLIDE 12

Dirichlet Problem

$$u_\Gamma(\vec{x}) = \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x}-\vec{x}'\|} \sigma(\vec{x}') d\Gamma' \quad \vec{x} \in \Gamma$$

Neumann Problem

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = \frac{\partial}{\partial n_{\vec{x}}} \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x}-\vec{x}'\|} \sigma(\vec{x}') d\Gamma' \quad \vec{x} \in \Gamma$$

Neumann Problem generates Hypersingular Integral

3.2.2 Dirichlet Problem 2nd Kind!

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$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = 2\pi\sigma(\vec{x}') + \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x}-\vec{x}'\|} \sigma(\vec{x}') d\Gamma'$$

3.2.3 Radiation Condition

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$$lim_{\|\vec{x}\|\to\infty} u(\vec{x}) = \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x}-\vec{x}'\|} \sigma(\vec{x}') d\Gamma' \to O(\|\vec{x}\|^{-2})$$

Add Extra Term to slow decay

$$u(\vec{x}) = \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x}-\vec{x}'\|} \sigma(\vec{x}') d\Gamma' + \alpha G(\vec{x}^*) \quad \vec{x}^* \ni \Omega$$

4 Green's Theorem Approach

4.1 Green's Second Identity

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$$\int_\Omega \left[u\nabla^2 w - w\nabla^2 u\right] d\Omega = \int_\Gamma \left[w\frac{\partial u}{\partial n} - u\frac{\partial w}{\partial n} d\Gamma\right]$$

Now let $w = \frac{1}{\|\vec{x}-\vec{x}'\|}$

$$2\pi u(\vec{x}) = \int_\Gamma \left[\frac{1}{\|\vec{x}-\vec{x}'\|}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x}-\vec{x}'\|} d\Gamma\right]$$

Easy to implement any boundary conditions!