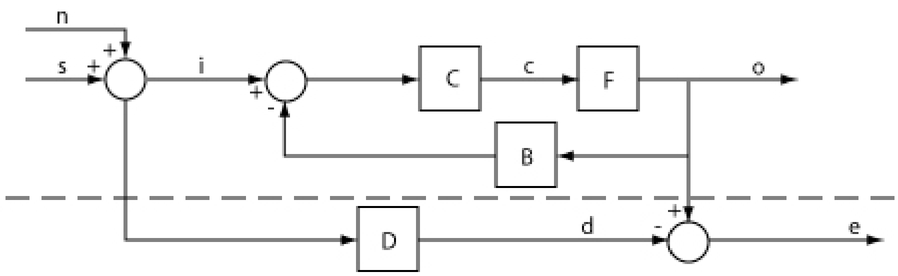
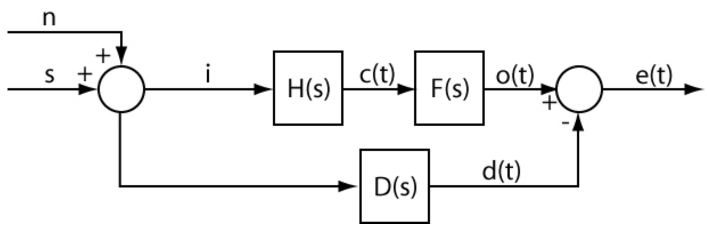
Lecture 18

Last time: Semi-free configuration design



This is equivalent to:



Note n, s enter the system at the same place. F is fixed. We design C (and perhaps B). We must stabilize F if it is given as unstable.

$$H(s) = \frac{C(s)}{1 + C(s)F(s)B(s)}$$

so that having the optimum H, we determine C from

$$C(s) = \frac{H(s)}{1 - H(s)F(s)B(s)}$$

We do not collect H and F together because if F is non-minimum phase, we would not wish to define H by

$$H = \frac{(HF)_{\text{opt}}}{F}$$

This leads to an unstable mode which is not observable at the output – thus cannot be controlled by feeding back.

Associate weighting functions with the given transfer functions.

$H(s) \rightarrow w_H(t)$

$F(s) \rightarrow w_F(t)$

$D(s) \rightarrow w_D(t)$

If $F(s)$ is unstable, put a stabilizing feedback around it, later associate it with the rest of the system.

Error Analysis

We require the mean squared error.

$$c(t) = \int_{-\infty}^{\infty} w_H(\tau_1) i(t-\tau_1) d\tau_1$$

$$o(t) = \int_{-\infty}^{\infty} w_F(\tau_2) c(t-\tau_2) d\tau_2$$

$$= \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_1 w_H(\tau_1) i(t-\tau_1-\tau_2)$$

$$d(t) = \int_{-\infty}^{\infty} w_D(\tau_3) s(t-\tau_3) d\tau_3$$

$$e(t) = o(t) - d(t)$$

$$\overline{e(t)^2} = \overline{o(t)^2} - 2\overline{o(t)d(t)} + \overline{d(t)^2}$$

$$\overline{o(t)^2} = \overline{\left[\int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_1 w_H(\tau_1) i(t-\tau_1-\tau_2) \right] \left[\int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) \int_{-\infty}^{\infty} d\tau_3 w_H(\tau_3) i(t-\tau_3-\tau_4) \right]}$$

$$= \int_{-\infty}^{\infty} d\tau_1 w_H(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_H(\tau_3) \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) \overline{i(t-\tau_1-\tau_2) i(t-\tau_3-\tau_4)}$$

$$= \int_{-\infty}^{\infty} d\tau_1 w_H(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_H(\tau_3) \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) R_{ii}(\tau_1+\tau_2-\tau_3-\tau_4)$$

$$\overline{o(t)d(t)} = \overline{\left[\int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_1 w_H(\tau_1) i(t-\tau_1-\tau_2) \right] \left[\int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) s(t-\tau_3) \right]}$$

$$= \int_{-\infty}^{\infty} d\tau_1 w_H(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) \overline{i(t-\tau_1-\tau_2) s(t-\tau_3)}$$

$$= \int_{-\infty}^{\infty} d\tau_1 w_H(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) R_{is}(\tau_1+\tau_2-\tau_3)$$

We shall not require $\overline{d(t)^2}$ in integral form.

The problem now is to choose $w_H(t)$ so as to minimize this $\overline{e(t)^2}$, for which we use variational calculus.

Let:

$$w_H(t) = w_0(t) + \delta w(t)$$

where $w_0(t)$ is the optimum weighting function (to be determined) and $\delta w(t)$ is an arbitrary variation – arbitrary except that it must be physically realizable.

Calculate the optimum $\overline{e^2}$ and its first and second variations.

$$\overline{e^2} = \overline{e_0^2} + \delta\overline{e^2} + \delta^2\overline{e^2}$$
$$\overline{e^2} = \overline{o(t)^2} + 2\overline{o(t)d(t)} + \overline{d(t)^2}$$

The optimum $\overline{e^2}$ ($\overline{e^2}$ for $\delta w(t) = 0$):

$$\overline{e(t)_0^2} = \int_{-\infty}^{\infty} d\tau_1 w_0(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_0(\tau_3) \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) R_{ii}(\tau_1 + \tau_2 - \tau_3 - \tau_4)$$
$$-2\int_{-\infty}^{\infty} d\tau_1 w_0(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) R_{is}(\tau_1 + \tau_2 - \tau_3) + \overline{d(t)^2}$$

The first variation in $\overline{e(t)^2}$ is

$$\delta\overline{e(t)^2} = \int_{-\infty}^{\infty} d\tau_1 \delta w(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_0(\tau_3) \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) R_{ii}(\tau_1 + \tau_2 - \tau_3 - \tau_4)$$
$$+\int_{-\infty}^{\infty} d\tau_1 w_0(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 \delta w(\tau_3) \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) R_{ii}(\tau_1 + \tau_2 - \tau_3 - \tau_4)$$
$$-2\int_{-\infty}^{\infty} d\tau_1 \delta w(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) R_{is}(\tau_1 + \tau_2 - \tau_3)$$

In the second term, let:

$$\tau_1 = \tau_3'$$
$$\tau_2 = \tau_4'$$
$$\tau_3 = \tau_1'$$
$$\tau_4 = \tau_2'$$

and interchange the order of integration.

2nd term $= \int_{-\infty}^{\infty} d\tau_1' \delta w(\tau_1') \int_{-\infty}^{\infty} d\tau_2' w_F(\tau_2') \int_{-\infty}^{\infty} d\tau_3' w_0'(\tau_3') \int_{-\infty}^{\infty} d\tau_4' w_F(\tau_4') R_{ii}(\tau_3' + \tau_4' - \tau_1' - \tau_2')$

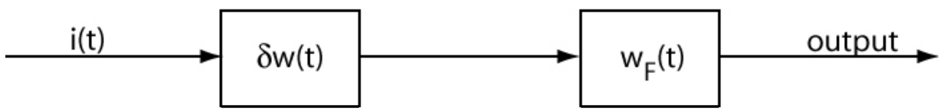
but since $R_{ii}(\tau_3' + \tau_4' - \tau_1' - \tau_2') = R_{ii}(\tau_1' + \tau_2' - \tau_3' - \tau_4')$ we see that the second term is exactly equal to the first term. Collecting these terms and separating out the common integral with respect to τ_1 gives

$$\delta\overline{e(t)^2} = 2\int_{-\infty}^{\infty} d\tau_1 \delta w(\tau_1) \left\{ \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_0(\tau_3) \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) R_{ii}(\tau_1 + \tau_2 - \tau_3 - \tau_4) \right.$$
$$\left. - \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) R_{is}(\tau_1 + \tau_2 - \tau_3) \right\}$$

The second variation of $\overline{e(t)^2}$ is

$$\delta^2\overline{e(t)^2} = \int_{-\infty}^{\infty} d\tau_1 \delta w(\tau_1) \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 \delta w(\tau_3) \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) R_{ii}(\tau_1 + \tau_2 - \tau_3 - \tau_4)$$

By comparison with the expression for $\overline{o(t)^2}$, this is seen to be the mean squared output of the system



$$\overline{(\text{output})^2} = \delta^2\overline{e(t)^2} > 0, \text{ non-zero input}$$

This second variation must be greater than zero, so the stationary point defined by the vanishing of the first variation is shown to be a minimum.

In the expression for the first variation, $\delta w(\tau_1) = 0$ for $\tau_1 < 0$ by the requirement that the variation be physically realizable. But $\delta w(\tau_1)$ is arbitrary for $\tau_1 \geq 0$, so we can be assured of the vanishing of $\delta\overline{e(t)^2}$ only if the $\{\ \}$ term vanishes almost everywhere for $\tau_1 \geq 0$. The condition which defines the minimum in $\overline{e(t)^2}$ is then

$$\int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_0(\tau_3) \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) R_{ii}(\tau_1 + \tau_2 - \tau_3 - \tau_4)$$
$$-\int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) R_{is}(\tau_1 + \tau_2 - \tau_3) = 0$$

for all τ_1, non-real-time.

Using this condition in the expression for $\overline{e(t)_0^2}$ and remembering that $w_0(t) = 0$ for $t < 0$ gives the result

$$\overline{e(t)_0^2} = \overline{d(t)^2} - \overline{o(t)_0^2}$$

which is convenient for the calculation of $\overline{e(t)_0^2}$.

Also since $\overline{o(t)_0^2} = \overline{d(t)^2} - \overline{e(t)_0^2}$, this says the optimum mean squared output is always less than the mean squared desired output.

Autocorrelation Functions

We have arrived at an extended form of the Wiener-Kopf equation which defines the optimum linear system under the ground rules stated before.

Recall that:

$$R_{ii}(\tau) = R_{ss}(\tau) + R_{sn}(\tau) + R_{ns}(\tau) + R_{nn}(\tau)$$
$$R_{is}(\tau) = R_{ss}(\tau) + R_{ns}(\tau)$$

since $i = s + n$.

The free configuration problem is a specialization of the semi-free configuration. In this expression we would take $F(s) = 1$, or $w_F(t) = \delta(t)$. In that case we have

$$\int_{-\infty}^{\infty} d\tau_2 d(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_0(\tau_3) \int_{-\infty}^{\infty} d\tau_4 \delta(\tau_4) R_{ii}(\tau_1 + \tau_2 - \tau_3 - \tau_4)$$
$$- \int_{-\infty}^{\infty} d\tau_2 \delta(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) R_{is}(\tau_1 + \tau_2 - \tau_3) =$$
$$\int_{-\infty}^{\infty} w_0(\tau_3) R_{ii}(\tau_1 - \tau_3) d\tau_3 - \int_{-\infty}^{\infty} w_D(\tau_3) R_{is}(\tau_1 - \tau_3) d\tau_3 = 0 \text{ for } \tau_1 \geq 0$$