Constitutive relations: Fourier law of heat conduction

$$q_i = q_i(\nabla\theta)$$

Linear isotropic: $q_i = -k\,\theta_{,i}$

Weak formulation: Weighted residuals

$$\int_B \left(-\rho c\dot{\theta} + f + q_{i,i}\right)\eta \, dV = 0 \quad \forall \text{ admissible } \eta$$

weak form:

$$\boxed{\int_B \left[(\rho c\dot{\theta} - f)\eta + q_i\,\eta_{,i}\right] dV - \int_{S_2} \overbrace{q_i n_i}^{\bar{q}}\,\eta \, dS = 0 \quad (\eta \text{ admissible})}$$

Finite element discretization (spatial)

$$\theta_h \cong \sum_{a=1}^{N} \theta_a N_a = \sum_{e=1}^{E}\sum_{a=1}^{n} \theta_a^e N_a^e$$

$$\dot{\theta}_h = \sum_{a=1}^{N} \dot{\theta}_a N_a = \sum_{e=1}^{E}\sum_{a=1}^{n} \dot{\theta}_a^e N_a^e$$

Insert in weak form:

$$\sum_e \left\{ \int_{\Omega^e} \left[\left(\rho^e c^e \sum_{b=1}^{n} \dot{\theta}_b N_b - f \right) \sum_{a=1}^{n} \eta_a N_a + q_i^e(\theta_h) \sum_{a=1}^{n} \eta_a N^e_{a,i} \right] dV - \int_{\Omega^e} f \sum_{a=1}^{n} \eta_a N^e_a \, dV - \int_{\partial\Omega^e \cap S_2} \bar{q}^e \sum_{a=1}^{n} \eta_a N^e_a \, dS \right\} = 0$$

$$\sum_e \sum_{a=1}^{n} \eta_a \left[\sum_{b=1}^{n} \Bigl(\overbrace{\int_{\Omega^e} \rho^e c^e N^e_a N^e_b \, dV}^{C^e} \Bigr) \dot{\theta}_b + \overbrace{\int_{\Omega^e} q_i^e(\theta_h) N^e_{a,i} \, dV}^{f^e_{int}} \underbrace{- \int_{\Omega^e} f^e N^e_a \, dV - \int_{\partial\Omega^e \cap S_2} \bar{q}^e N^e_a \, dS}_{f^e_{ext}} \right] = 0$$

$$\Rightarrow \boxed{C\dot{\theta} + f^{int}(\theta) = f^{ext}} \quad \text{where}$$

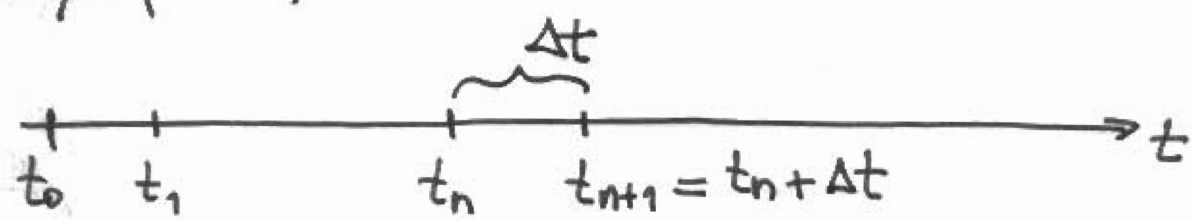
$$\boxed{C = \sum_e C^e = \sum_e \int_{\Omega^e} \rho^e c^e N^e_a N^e_b \, dV}$$

$$\boxed{f^{int} = \sum_e f^e_{int} = \sum_e \int_{\Omega^e} P^e_i(t_n)\, N^e_{a,i}\, dV}$$

$$\boxed{f^{ext} = \sum_e f^e_{ext} = \sum_e \left\{ \int_{\Omega^e} f^e N^e_a\, dV + \int_{\partial\Omega^e \cap S_2} \bar{P}^e N_a\, dS \right\}}$$

Time-stepping algorithms

Envision incremental solution procedure: Given $x_0, v_0, f^{ext}(t)$; and an increasing sequence of (evenly spaced) discrete times:



we wish to determine

$$\begin{Bmatrix} x_0 \\ v_0 \end{Bmatrix} \boxed{\begin{Bmatrix} x_1 \\ v_1 \end{Bmatrix} \cdots\cdots \begin{Bmatrix} x_n \\ v_n \end{Bmatrix} \begin{Bmatrix} x_{n+1} \\ v_{n+1} \end{Bmatrix}}$$

We need some scheme to march in time:

$$\begin{Bmatrix} x_n \\ v_n \end{Bmatrix} \xrightarrow{\text{ALGORITHM}} \begin{Bmatrix} x_{n+1} \\ v_{n+1} \end{Bmatrix}$$

(one-step formuli)

Example: <u>Newmark algorithm</u>

- $x_{n+1} = x_n + \Delta t\, v_n + \Delta t^2 \left[\left(\frac{1}{2} - \beta \right) a_n + \beta\, a_{n+1} \right]$
- $v_{n+1} = v_n + \Delta t \left[(1-\gamma)\, a_n + \gamma\, a_{n+1} \right]$

where a_n; a_{n+1} follow from

$M a_n + f^{int}(x_n, v_n) = f^{ext}(t_n)$

- $M a_{n+1} + f^{int}(x_{n+1}, v_{n+1}) = f^{ext}(t_{n+1})$

$v_n = \dot{x}_n \;,\; a_n = \ddot{x}_n \;,\; f_n^{ext} = f^{ext}(t_n)$

- β, γ: Newmark parameters

$0 \leqslant \beta \leqslant 0.5 \qquad 0 \leqslant \gamma \leqslant 1$

Red-dot equations define a set of nonlinear algebraic equations on $(x_{n+1}, v_{n+1}, a_{n+1})$ as a function of (x_n, v_n, a_n).

Nonlinear system solved by Newton-Raphson iteration:

$$\begin{Bmatrix} x_{n+1}^{(0)} \\ v_{n+1}^{(0)} \end{Bmatrix}, \begin{Bmatrix} x_{n+1}^{(1)} \\ v_{n+1}^{(1)} \end{Bmatrix} \cdots\cdots \begin{Bmatrix} x_{n+1}^{(k)} \\ v_{n+1}^{(k)} \end{Bmatrix} \begin{Bmatrix} x_{n+1}^{(k+1)} \\ v_{n+1}^{(k+1)} \end{Bmatrix} \cdots$$

$$\uparrow \begin{Bmatrix} x_n \\ v_n \end{Bmatrix} \qquad\qquad \uparrow \begin{Bmatrix} x_{n+1} \\ v_{n+1} \end{Bmatrix}$$

(convergence)

<u>Solution procedure</u> (not unique)

1) <u>Newmark predictors</u> (retain all the explicit terms as first guess)

$$\begin{cases} x_{n+1}^{(0)} = x_n + \Delta t\, v_n + \Delta t^2 \left(\frac{1}{2} - \beta\right) a_n \\ v_{n+1}^{(0)} = v_n + \Delta t\,(1-\gamma)\, a_n \\ a_{n+1}^{(0)} = 0 \end{cases}$$

2) Know $\left\{ x_{n+1}^{(k)}, v_{n+1}^{(k)}, a_{n+1}^{(k)} \right\}$. Linearize about it:

$$\begin{cases} x_{n+1}^{(k+1)} = x_{n+1}^{(k)} + \Delta x \\ v_{n+1}^{(k+1)} = v_{n+1}^{(k)} + \Delta v \\ a_{n+1}^{(k+1)} = a_{n+1}^{(k)} + \Delta a \end{cases}$$

The first two equations are trivial since they are linear:

$$\begin{cases} ① \; x_{n+1}^{(k+1)} = x_{n+1}^{(k)} + \overbrace{\beta \Delta t^2 \Delta a}^{\Delta x} \\ ② \; v_{n+1}^{(k+1)} = v_{n+1}^{(k)} + \underbrace{\gamma \Delta t \, \Delta a}_{\Delta v} \end{cases}$$

$$M\left(a_{n+1}^{(k)} + \Delta a\right) + f^{int}\left(x_{n+1}^{(k)} + \Delta x, v_{n+1}^{(k)} + \Delta v\right) = f_{n+1}^{ext}$$

$$M\left(a_{n+1}^{(k)} + \Delta a\right) + f^{int}\left(x_{n+1}^{(k)}, v_{n+1}^{(k)}\right) +$$

$$+ \underbrace{\left.\frac{\partial f^{int}}{\partial x}\right|_{(x_{n+1}^{k}, v_{n+1}^{k})}}_{K_{n+1}^{k} = K(x_{n+1}^{k}, v_{n+1}^{k})} \Delta x + \underbrace{\left.\frac{\partial f^{int}}{\partial v}\right|_{(x_{n+1}^{k}, v_{n+1}^{k})}}_{C_{n+1}^{k}(x_{n+1}^{k}, v_{n+1}^{k})} \Delta v \sim f_{n+1}^{ext}$$

$$M \Delta a + K_{n+1}^{k} \Delta x + C_{n+1}^{k} \Delta v = f_{n+1}^{ext} - M a_{n+1}^{(k)} - f^{int}(x_{n+1}^{k}, v_{n+1}^{k})$$

Solve ① and ② explicitly:

$$\Delta a = \frac{\Delta x}{\beta \Delta t^2} \qquad \Delta v = \gamma \Delta t \, \Delta a = \frac{\gamma}{\beta \Delta t} \Delta x$$

$$\underbrace{\left(\frac{1}{\beta \Delta t^2} M + K_{n+1}^{k} + \frac{\gamma}{\beta \Delta t} C_{n+1}^{k} \right)}_{(K^{eff})_{n+1}^{k}} \Delta x = f_{n+1}^{ext} - f^{int}_{(x_{n+1}^{k}, v_{n+1}^{k})} - M a_{n+1}^{(k)}$$

$$\boxed{(K^{eff})_{n+1}^{k} \Delta x = r_{n+1}}$$

$$K^{eff} = K + \frac{\gamma}{\beta \Delta t} C + \frac{1}{\beta \Delta t^2} M$$

$$r = f^{ext} - f^{int} - Ma$$

③ Newmark correctors

- $x_{n+1}^{(k+1)} = x_{n+1}^{(k)} + \Delta x$
- $v_{n+1}^{(k+1)} = v_{n+1}^{(k)} + \dfrac{\gamma}{\beta \Delta t} \Delta x$

- $a_{n+1}^{(k+1)} = a_{n+1}^{(k)} + \frac{1}{\beta \Delta t^2} \Delta x$

4) Convergence check:

$$\| r_{n+1}^{(k+1)} \| \leq TOL \, \| r_{n+1}^{(0)} \| \,?\ \text{EXIT}: k \leftarrow k+1 \ \text{GOTO} \ 2$$

5) $n \leftarrow n+1$ until $t_{n+1} = t_{max}$

Newmark is <u>implicit</u> (implies equation solving) for all values of (β, γ) except for especial case of $\beta = 0$, no damping: $f^{int} = f^{int}(x)$

$\longrightarrow$ <u>explicit dynamics</u>

- $\beta = 0$ ① $x_{n+1} = x_n + \Delta t \, v_n + \frac{\Delta t^2}{2} a_n$

③ $M a_{n+1} + f^{int}(x_{n+1}) = f_{n+1}^{ext}$

$$\boxed{a_{n+1} = M^{-1} \left(f_{n+1}^{ext} - f^{int}(x_{n+1}) \right)}$$

No equation solving if "M" is diagonal