Energy conservation/dissipation

$$\frac{d}{dt}\|y\|_E^2 = \frac{d}{dt}(y^T A y) \underset{\uparrow A=A^T}{=} 2y^T A \dot{y} = 2y^T(b - By)$$

$$\frac{1}{2}\frac{d}{dt}\|y\|_E^2 = y^T(b-By) = b^T y - y^T B y$$

Assume system is autonomous (unforced, isolated)

$$b=0 \Longrightarrow \frac{1}{2}\frac{d}{dt}\|y\|_E^2 = -y^T B y$$

Decompose $B = \text{sym}\,B + \text{skew}\,B$

$$\text{sym}\,B = \frac{1}{2}(B + B^T), \quad \text{skew}\,B = \frac{1}{2}(B - B^T)$$

$$\Longrightarrow \boxed{\frac{1}{2}\frac{d}{dt}\|y\|_E^2 = -y^T \text{sym}\,B\, y}$$

$\boxed{\text{System is conservative iff } B^T = -B \text{ (purely skew)}}$

Example: • Dynamics: $B = \begin{pmatrix} 0 & -K \\ K & C \end{pmatrix}$

Assume $C = C^T$

$$\operatorname{sym} B = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \qquad \operatorname{skew} B = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix}$$

$$\Longrightarrow \underline{C = 0} \iff B^T = -B \iff \underline{\text{conservative system}}$$

<u>Dissipative systems</u>: $\operatorname{sym} B > 0$

- Dynamics: $\operatorname{sym} B > 0 \iff C > 0$
- Heat conduction: $\operatorname{sym} B > 0 \iff K > 0$

- <u>Abstract algorithms</u>: Linear dependence between y_{n+1} and y_n

n	$n+1$
t_n	t_{n+1}
y_n	y_{n+1}

consider <u>linear</u> <u>unforced</u> case $\Longrightarrow$

algorithm: $y_n \rightarrow y_{n+1}$ is a linear mapping

$$\boxed{y_{n+1} = \underline{F(\Delta t)}\, y_n}$$

$\rightarrow$ Amplification matrix of the algorithm

One-to-one correspondence between linear algorithms and amplification matrices

Example: • Trapezoidal rule:

$$A \frac{y_{n+1} - y_n}{\Delta t} + B\left[(1-\alpha) y_n + \alpha y_{n+1}\right] = 0$$

$$\underbrace{(A + \alpha B \Delta t)}_{P(\Delta t)} y_{n+1} = \underbrace{(A - B(1-\alpha)\Delta t)}_{Q(\Delta t)} y_n$$

$$\boxed{F(\Delta t) = P^{-1}(\Delta t)\, Q(\Delta t)}$$

$$F = (A + \alpha \Delta t B)^{-1} (A - (1-\alpha)\Delta t B)$$

• Newmark

$$\begin{cases} ① \; x_{n+1} = x_n + \Delta t\, v_n + \Delta t^2\left[(1/2 - \beta) a_n + \beta a_{n+1}\right] \\ ② \; v_{n+1} = v_n + \Delta t\left[(1-\gamma) a_n + \gamma a_{n+1}\right] \\ M a_{n+1} + C v_{n+1} + K x = 0 \end{cases}$$

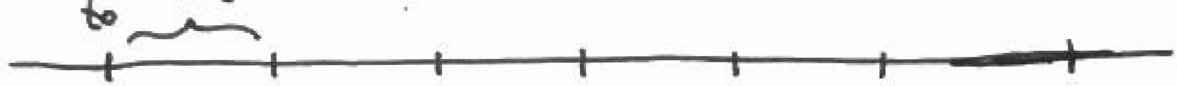
$$x_{n+1} = x_n + \Delta t\, v_n + \Delta t^2 \Big[\left(\tfrac{1}{2} - \beta\right)\left(-M^{-1}\right)\left(C v_n + K x_n\right) + \beta\left(-M^{-1}\right)\left(C v_{n+1} + K x_{n+1}\right) \Big]$$

$$v_{n+1} = v_n + \Delta t \Big[(1-\gamma)\left(-M^{-1}\right)\left(C v_n + K x_n\right) + \gamma\left(-M^{-1}\right)\left(C v_{n+1} + K x_{n+1}\right) \Big]$$

$$\underbrace{\begin{pmatrix} M + \beta \Delta t^2 K & +\beta \Delta t^2 C \\ \gamma \Delta t K & M + \gamma \Delta t C \end{pmatrix}}_{P} \begin{Bmatrix} x_{n+1} \\ v_{n+1} \end{Bmatrix} = \underbrace{\begin{pmatrix} M - \Delta t^2 \left(\tfrac{1}{2} - \beta\right) K & \Delta t M - \left(\tfrac{1}{2} - \beta\right)\Delta t^2 C \\ -\Delta t (1-\gamma) K & M - \Delta t (1-\gamma) C \end{pmatrix}}_{Q} \begin{Bmatrix} x_n \\ v_n \end{Bmatrix}$$

Convergence:

Under what conditions is the time stepping algorithm convergent?



t_0, Δt

y_0

$y_1 = F(\Delta t)\, y_0$

$y_2 = F(\Delta t)\, y_1 = F^2(\Delta t)\, y_0$

$y_n = F^n(\Delta t)\, y_0$

Convergence: Fix $t \Rightarrow \Delta t = \frac{t}{n}$

$$\boxed{\lim_{n\to\infty} F^n\left(\frac{t}{n}\right) y_0 = y(t)}$$

$y(t)$: exact solution

$\forall y_0$

Exact solution: Want to get rid of initial condition in expression of convergence

Definition: Exponential mapping of square matrices

$$e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k = I + M + \frac{M^2}{2} + \cdots$$

$$\frac{d}{dt} e^{tM} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{1}{k!} M^k t^k\right) = \sum_{k=1}^{\infty} \frac{k}{k!} t^{(k-1)} M^k$$

$$= \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} M^k = M \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} M^{(k-1)}$$

$$= M e^{tM}$$

Apply to: $A\dot{y} + By = 0$, $y(0) = y_0$

Claim $y(t) = e^{Mt} y_0$

$$\dot{y}(t) = M e^{Mt} y_0$$

$$AMe^{Mt} y_0 + B e^{Mt} y_0 = 0 \implies M = -A^{-1}B$$

$$y(t) = e^{-A^{-1}Bt} y_0$$

$\implies$ convergence can be written as:

$$\lim_{n \to \infty} F^n\left(\frac{t}{n}\right) \cancel{y_0} = e^{-A^{-1}Bt} \cancel{y_0} \quad \forall y_0$$

$$\boxed{\lim_{n \to \infty} F^n\left(\frac{t}{n}\right) = e^{-A^{-1}Bt}}$$

$$F(\Delta t) = \underbrace{P^{-1}(\Delta t)}_{\text{implicit part}} \; \underbrace{Q(\Delta t)}_{\text{explicit part}}$$

<u>Conditions for convergence:</u>

Lax equivalence theorem:
consistency + stability $\iff$ convergence

Consistency

$$A\dot{y} + By = 0 \ ;$$

$$y_{n+1} = F(\Delta t)\, y_n$$

$$F(\Delta t)\, y_n = y(t_n) + \Delta t\, \dot{y}(t_n) + O(\Delta t^2) \quad \forall y(t_n)$$

The exact rate: $\dot{y}(t_n) = -A^{-1}B\, y(t_n)$

$$F(\Delta t)\, y_n = y(t_n) + \Delta t \left(-A^{-1}B\, y(t_n)\right) + O(\Delta t^2)$$

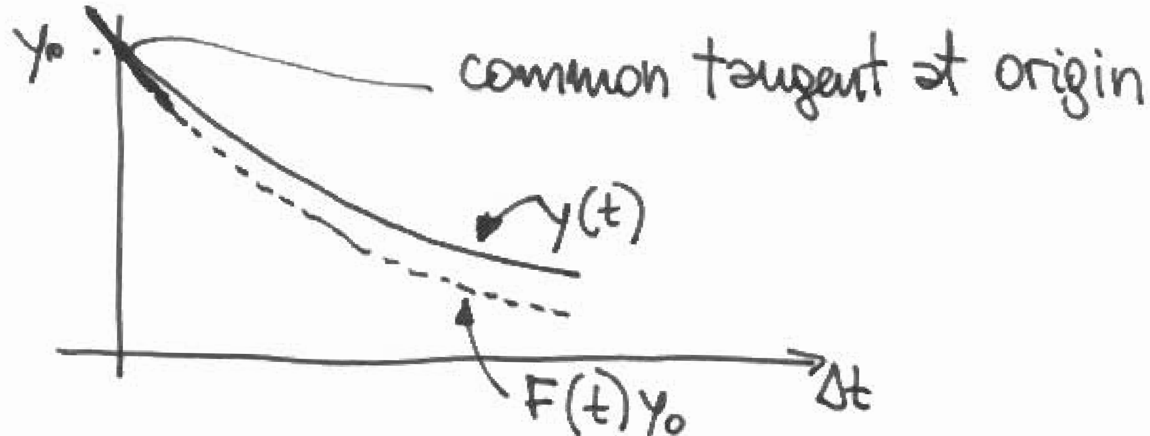
$$\left(\frac{F(\Delta t) - I}{\Delta t}\right) y(t_n) = -A^{-1}B\, y(t_n) + O(\Delta t^2) \quad \forall y(t_n)$$

$$\boxed{\left[\lim_{\Delta t \to 0} \frac{F(\Delta t) - I}{\Delta t}\right] \cdot = -A^{-1}B} \quad \text{A}$$

Also, since $\lim_{\Delta t \to 0} F(\Delta t) = I$ or equivalently by L'Hôpital's rule

$$\boxed{\left.\frac{d}{d\Delta t} F(\Delta t)\right|_{\Delta t = 0} = -A^{-1}B} \quad \text{B}$$

An algorithm $F(\Delta t)$ is said to be consistent iff Ⓐ or Ⓑ hold.



Write consistency condition in terms of P, Q

$$F(\Delta t) = P^{-1}(\Delta t)\, Q(\Delta t)$$

consistency: $\left.\frac{d}{d\Delta t} F(\Delta t)\right|_{\Delta t=0} = -A^{-1}B$

$$F' = (P^{-1})' Q + P^{-1} Q'$$

$$= -P^{-1} P' \underbrace{P^{-1} Q}_{F} + P^{-1} Q'$$

$$F'(0) = -P^{-1}(0)\, P'(0) \underbrace{F(0)}_{I} + P^{-1}(0)\, Q'(0)$$

$$F(0) = I = P^{-1}(0)\, Q(0) \implies P(0) = Q(0)$$

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$$\boxed{-P^{-1}(0)\,P'(0) + Q^{-1}(0)\,Q'(0) = -A^{-1}B}$$