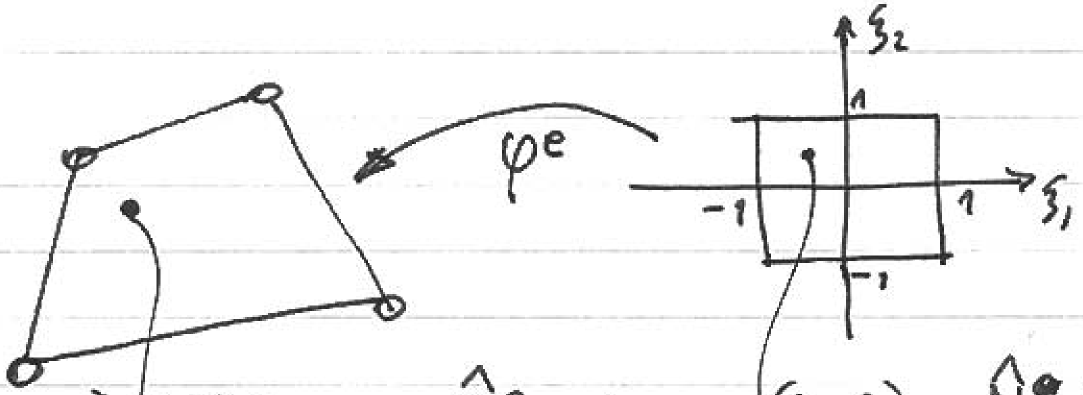
$$x_i(\xi_\alpha) = \varphi_i^e(\xi_\alpha) = \sum_{a=1}^{n} x_{ia}^e \hat{N}_a(\xi_\alpha)$$



$N_a^e(x_1, x_2) \equiv \hat{N}_a^e(\xi_1, \xi_2)$ $\qquad$ $(\xi_1, \xi_2) \to \hat{N}_a^e(\xi_1, \xi_2)$

$$\boxed{N_a^e(x) = \hat{N}_a(\underbrace{\varphi^{e^{-1}}(x)}_{\xi}) = \hat{N}_a \circ \varphi^{-1}}$$

Need to compute $\dfrac{\partial N_a^e}{\partial x_i} = \dfrac{\partial \hat{N}_a(\varphi^{e^{-1}}(x))}{\partial \xi_\alpha} \dfrac{\partial (\varphi^{e^{-1}})_\alpha}{\partial x_i}$

Jacobian matrix: $\underbrace{J_{i\alpha}^e}_{} = \dfrac{\partial x_i}{\partial \xi_\alpha} = \sum_{a=1}^{n} x_{ia} \dfrac{\partial \hat{N}_a}{\partial \xi_\alpha}$

$\dim(J_{i\alpha}^e) = 2 \times 2$

$\dfrac{\partial (\varphi^e)_\alpha^{-1}}{\partial x_i} = (J_{\alpha i}^e)^{-1}$ invert 2x2 matrix!

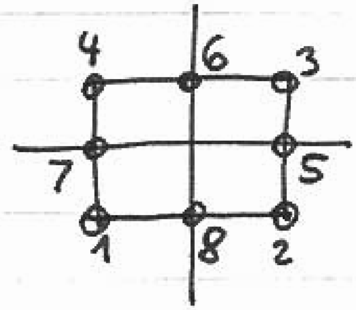
$$\frac{\partial N_a^e}{\partial x_i} = \frac{\partial \hat{N}_a}{\partial \xi_\alpha} (J_{\alpha i}^e)^{-1}$$

shp(i, a)

store: $N_{a,i} \rightarrow shp(i,a)$
$\hat{N}_a(\xi) = N_a(x) \rightarrow shp(d+1,a)$

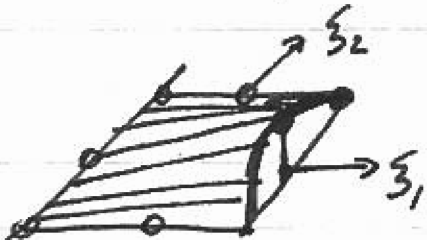
Higher order interpolation

8-node quadrilateral



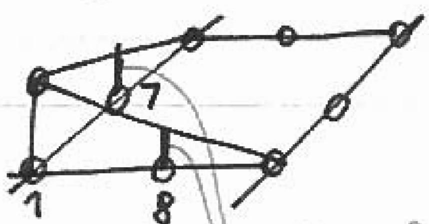
Obtain by construction:

$$\hat{N}_5(\xi_1,\xi_2) = \frac{(1+\xi_1)}{2}(1-\xi_2)(1+\xi_2)$$



similarly $\hat{N}_6, \hat{N}_7, \hat{N}_8$ (midnodes)

What about $\hat{N}_1$?

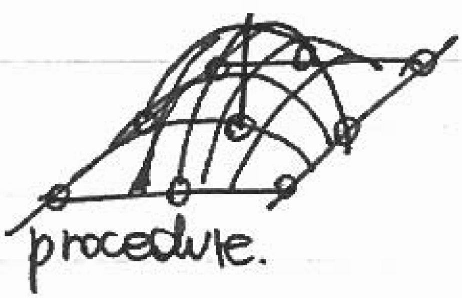


$\neq 0$ $\left(= \frac{1}{2}, \text{actually}\right)$

$$\hat{N}_1^{new} = \hat{N}_1^{linear} - \frac{1}{2} N_8 - \frac{1}{2} N_7$$

$\vdots$ $\hat{N}_2^{new}$, $\hat{N}_3^{new}$, $\hat{N}_4^{new}$

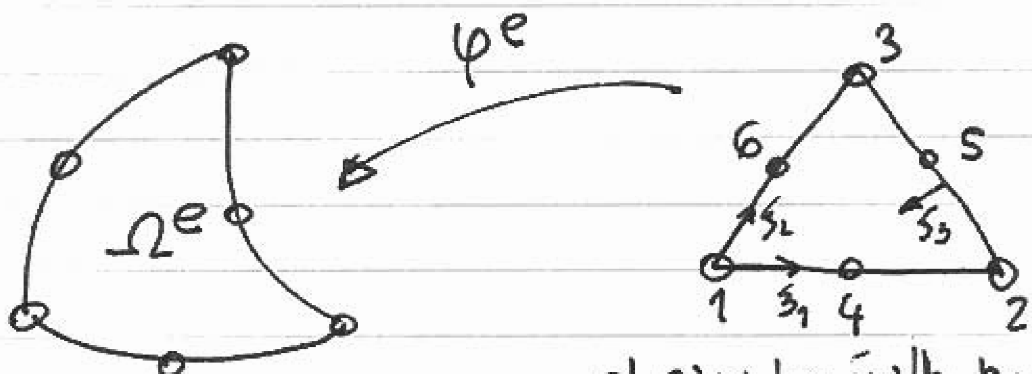
Want to add 9th node?



$\hat{N}_9 = (1-\xi_1^2)(1-\xi_2^2)$ follow same procedure.

can implement variable node element all in one subroutine.

Isoparametric triangular elements



d-simplex with p_k interpolation

ξ_1, ξ_2, ξ_3 : baricentric coordinates

$$\xi_1 + \xi_2 + \xi_3 = 1$$

3-noded:

- $\hat{N}_1 = \xi_3 = 1 - \xi_1 - \xi_2$
- $\hat{N}_2 = \xi_1$
- $\hat{N}_3 = \xi_2$

6-noded:

$$\hat{N}_4 = 4\,\xi_1 \xi_3 \qquad \hat{N}_1^{6-N} = \hat{N}_1^{3-N} - \frac{1}{2}\hat{N}_4 - \frac{1}{2}\hat{N}_6$$

$$\hat{N}_5 = 4\,\xi_1 \xi_2 \qquad \hat{N}_2^{6-N} = \hat{N}_2^{3-N} - \frac{1}{2}\hat{N}_4 - \frac{1}{2}\hat{N}_5$$

$$\hat{N}_6 = 4\,\xi_2 \xi_3 \qquad \hat{N}_3^{6-N} = \hat{N}_3^{3-N} - \frac{1}{2}\hat{N}_5 - \frac{1}{2}\hat{N}_6$$

- $x_i^e = \varphi_i^e(\xi_j) = \sum_{a=1}^{n} X_{ia}^e \hat{N}_a(\xi_j) \qquad j=1,2,3$
- $\sum_j \xi_j = 1 \qquad j = 1, \dots, d+1$

$$N_a^e(x) = \hat{N}_a(\xi) = \hat{N}_a(\varphi^{-1}(x)) = (\hat{N}_a \circ \varphi^{-1})(x)$$

$\dfrac{\partial N_a^e}{\partial x_i}$? $\qquad \dfrac{\partial N_a^e}{\partial x_i} dx_i = \dfrac{\partial \hat{N}_a}{\partial \xi_\alpha} d\xi_\alpha$

$$\underbrace{dx_i}_{d \times 1} = \underbrace{\sum_{a=1}^{n} X_{ia}^e \frac{\partial \hat{N}_a}{\partial \xi_\alpha}}_{d \times (d+1)} \, d\xi_\alpha \qquad (d+1) \times 1$$

$$\frac{dx_i}{d\xi_\alpha}$$

But

$$\sum_\alpha \xi_\alpha = 1 \;\rightarrow\; d\xi_1 + d\xi_2 + d\xi_3 = 0$$

Can write equation:

$$\begin{Bmatrix} dx_i \\ \hline 0 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{dx_i}{d\xi_\alpha} \\ \hline 1 \quad 1 \quad 1 \end{bmatrix}}_{J_{i\alpha} \in \mathbb{R}^{(d+1) \times (d+1)}} \begin{Bmatrix} d\xi_1 \\ d\xi_2 \\ d\xi_3 \end{Bmatrix}$$

$$dx_i = J_{i\alpha}\, d\xi_\alpha \longrightarrow d\xi_\alpha = J^{-1}_{\alpha i}\, dx_i$$

$$\longrightarrow \frac{\partial N^e_a}{\partial x_i} = \frac{\partial \hat{N}_a}{\partial \xi_\alpha} J^{-1}_{\alpha i}$$

Numerical integration

Need to compute integrals:

$$K^e_{iakb} = \int_{\Omega^e} C_{ijkl}\, N_{a,j}\, N_{b,l}\, dV \quad , \text{ etc.}$$

representative $I = \int_{\Omega^e} f(x)\, dV$

isoparametric →

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi_1, \xi_2)\, J(\xi_1, \xi_2)\, d\xi_1\, d\xi_2$$

(representative integral)

$$J(\xi_1, \xi_2) = \det(J_{i\alpha})$$

- Seek "n"-point approximation of 1-D integral

$$I \sim \sum_{q=1}^{Q} w_q\, f(\xi_i) = I_Q(f)$$

w_q: weights
ξ_q: Gauss sampling points

Gauss quadrature: select the "Q" sample points "ξ_q" and weights "w_q" so that the rule is exact for the polynomial of highest order possible

- One-point formula (Q=1)

$I_1(f) = w_1 f(\xi_1)$, unknowns ξ_1, w_1

Should be able to integrate exactly a polynomial with two parameters, i.e., a linear function

$f(\xi) = a_0 + a_1 \xi$

$$\rightarrow \int_{-1}^{1} f(\xi)\,d\xi = 2a_0 + \cancel{\tfrac{2}{3}a_1}^{\,0} = w_1 (a_0 + a_1 \xi_1)$$

$$\Longleftrightarrow \quad w_1 = 2\,,\ \xi_1 = 0$$

$$\int_{-1}^{1} f(\xi)\,d\xi \approx 2 f(0)$$

- Two-point formula

$$I_2(f) = w_1 f(\xi_1) + w_2 f(\xi_2) = \int_{-1}^{1} a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 d\xi$$

$$(w_1+w_2)a_0 + (w_1\xi_1 + w_2\xi_2)a_1 + (w_1\xi_1^2 + w_2\xi_2^2)a_2 + (w_1\xi_1^3 + w_2\xi_2^3)a_3 = 2a_0 + \frac{2}{3}a_2$$

$$\Leftrightarrow \left.\begin{aligned} w_1 + w_2 &= 2 \\ w_1\xi_1 + w_2\xi_2 &= 0 \\ w_1\xi_1^2 + w_2\xi_2^2 &= \frac{2}{3} \\ w_1\xi_1^3 + w_2\xi_2^3 &= 0 \end{aligned}\right\} \quad \begin{aligned} w_1 &= w_2 = 1 \\ \xi_{1,2} &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

$$\Longrightarrow I_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Example: $f(\xi) = \cos(\xi)$

$$I = \int_{-1}^{1} \cos(\xi)\,d\xi = -\sin\xi\Big|_{-1}^{1} = 2\sin 1 = 1.68$$

$$I_1 = 2\cos(0) = 2$$

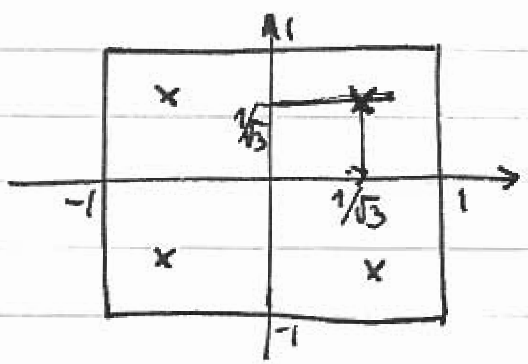
$$I_2 = \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 1.676$$

Two-dimensional integrals:

$$I(f) = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi\, d\eta$$

$$= \int_{-1}^{1} \sum_{q=1}^{Q} w_q\, f(\xi_q,\eta)\, d\eta$$

$$= \sum_{p=1}^{Q}\sum_{q=1}^{Q} w_p\, w_q\, f(\xi_q,\eta_p) = I_{p,q}$$



Gauss quadrature: $\underline{\underline{Q}}$ evaluations, Q known weights

integrates exactly polynomial with 2Q parameters

i.e., order $\underline{\underline{2Q-1}}$