Unsteady Local Scaling Transformation

$$u = \frac{\partial \psi}{\partial y} \qquad s = \nu \frac{\partial u}{\partial y} \qquad u_e = u_e(x,t)$$

$$\frac{\partial u}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} + \frac{\partial s}{\partial y}$$

Coordinate transformation: $(x, y, t) \rightarrow (\xi, \eta, \tau)$

$$\xi = x \qquad \eta = y/\Delta(x,t) \qquad \tau = t$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \xi} - \frac{\eta}{\Delta}\frac{\partial \Delta}{\partial \xi}\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} + \frac{\partial \tau}{\partial y}\frac{\partial}{\partial \tau} = \frac{1}{\Delta}\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau} - \frac{\eta}{\Delta}\frac{\partial \Delta}{\partial \tau}\frac{\partial}{\partial \eta}$$

Variable transformation: $(\psi \;\; u \;\; s \;\; u_e) \rightarrow (F \;\; U \;\; S \;\; u_e)$

$$\psi = nF \qquad u = u_e U \qquad s = \frac{\Delta}{\xi} u_e^2 S \qquad (n \equiv u_e \Delta)$$

$$\frac{\partial S}{\partial \eta} + \frac{\xi}{n}\frac{\partial n}{\partial \xi} F \frac{\partial U}{\partial \eta} + \frac{\xi}{u_e}\frac{\partial u_e}{\partial \xi}\left(1 - U\frac{\partial F}{\partial \eta}\right) + \eta \frac{\xi}{n}\frac{\partial \Delta}{\partial \tau}\frac{\partial U}{\partial \eta} + \frac{\xi}{u_e^2}\frac{\partial u_e}{\partial \tau}(1 - U) = \xi\left[\frac{\partial F}{\partial \eta}\frac{\partial U}{\partial \xi} - \frac{\partial F}{\partial \xi}\frac{\partial U}{\partial \eta} + \frac{1}{u_e}\frac{\partial U}{\partial \tau}\right]$$

IC's $(\tau = 0^+)$:
$$F(\xi, \eta, 0^+) = F_0(\xi, \eta)$$
$$U(\xi, \eta, 0^+) = U_0(\xi, \eta)$$
$$S(\xi, \eta, 0^+) = S_0(\xi, \eta)$$

BC's $(\tau > 0)$:
$$F(\xi, 0, \tau) = 0$$
$$U(\xi, 0, \tau) = 0$$
$$U(\xi, \eta_e, \tau) = 1$$

$u_e(\xi, \tau)$ specified
$\Delta(\xi, \tau)$ arbitrary

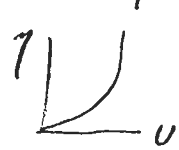
IMPULSIVE START: τ small. $u_e(\xi, 0^+)$ finite

$\Rightarrow$ choose $\Delta = \sqrt{\nu \tau}$

Equations reduce to $\dfrac{\partial^2 U}{\partial \eta^2} + \dfrac{\eta}{2}\dfrac{\partial U}{\partial \eta} = 0$

Rayleigh problem!
(except that freestream is started & wall is fixed)

$\Rightarrow U = \operatorname{erf}\left(\frac{\eta}{2}\right)$



Flow is initially potential (no separation). Viscous effects restricted to region within $y \sim \sqrt{\nu t}$ no matter what $u_e(\xi, t=0^+)$ is!

SIMILARITY VARIABLE DERIVATION   Ref: Cebeci + Bradshaw 86-90

Seek: TSL variable transformations $(x, y, u, v, u_e) \to (\xi, \eta, \hat{f}, \hat{g}, \hat{u}_e)$ of the form

$$\xi = x \qquad \eta = \frac{y}{x^{\alpha}} \qquad \hat{f} = \frac{u}{x^{\alpha_3}} \qquad \hat{g} = \frac{v}{x^{\alpha_4}} \qquad \hat{u}_e = \frac{u_e}{x^{\alpha_5}}$$

such that ξ-dependence in the transformed TSL equations drops out.
Using the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \alpha \frac{\eta}{\xi}\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{1}{\xi^{\alpha}}\frac{\partial}{\partial \eta} \qquad ; \quad \frac{\partial^2}{\partial y^2} = \frac{1}{\xi^{2\alpha}}\frac{\partial^2}{\partial \eta^2}$$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \alpha_3 \xi^{\alpha_3 - 1}\hat{f} + \xi^{\alpha_3}\left[\frac{\partial \hat{f}}{\partial \xi} - \alpha\frac{\eta}{\xi}\frac{\partial \hat{f}}{\partial \eta}\right] + \xi^{\alpha_4 - \alpha}\frac{\partial \hat{g}}{\partial \eta} = 0$$

$$\text{or} \quad \xi^{\alpha_3 - 1}\left\{\alpha_3\hat{f} + \xi\frac{\partial \hat{f}}{\partial \xi} - \alpha\eta\frac{\partial \hat{f}}{\partial \eta}\right\} + \xi^{\alpha_4 - \alpha}\left\{\frac{\partial \hat{g}}{\partial \eta}\right\} = 0$$

If this equation is to be independent of ξ, we must have

$$\alpha_3 - 1 = \alpha_4 - \alpha \qquad (1)$$

x - Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - u_e\frac{du_e}{dx} - \nu\frac{\partial^2 u}{\partial y^2} = x^{\alpha_3}\hat{f}\left[\alpha_3\xi^{\alpha_3-1}\hat{f} + \xi^{\alpha_3}\left[\frac{\partial \hat{f}}{\partial \xi} - \alpha\frac{\eta}{\xi}\frac{\partial \hat{f}}{\partial \eta}\right]\right]$$

$$+ \xi^{\alpha_4}\hat{g}\left[\xi^{\alpha_3-\alpha}\frac{\partial \hat{f}}{\partial \eta}\right] - \xi^{\alpha_5}\hat{u}_e\left[\alpha_5\xi^{\alpha_5-1}\hat{u}_e + \xi^{\alpha_5}\frac{d\hat{u}_e}{d\xi}\right] - \nu\xi^{\alpha_3-2\alpha}\frac{\partial^2 \hat{f}}{\partial \eta^2} = 0$$

$$\text{or} \quad \xi^{2\alpha_3-1}\left\{\hat{f}\left[\alpha_3\hat{f} + \xi\frac{\partial \hat{f}}{\partial \xi} - \alpha\eta\frac{\partial \hat{f}}{\partial \eta}\right]\right\} + \xi^{\alpha_4+\alpha_3-\alpha}\left\{\hat{g}\frac{\partial \hat{f}}{\partial \eta}\right\}$$

$$- \xi^{2\alpha_5-1}\left\{\hat{u}_e\left[\alpha_5\hat{u}_e + \xi\frac{d\hat{u}_e}{d\xi}\right]\right\} - \xi^{\alpha_3-2\alpha}\left\{\nu\frac{\partial^2 \hat{f}}{\partial \eta^2}\right\} = 0$$

Hence, we must have $\quad 2\alpha_3 - 1 = \alpha_4 + \alpha_3 - \alpha = 2\alpha_5 - 1 = \alpha_3 - 2\alpha \qquad (2)$

Also, $\hat{u}_e$ must be constant $\rightarrow u_e \sim x^m$, $m = 1 - 2\alpha$

Solution to (1) + (2) is $\quad \alpha_3 = \alpha_5 = 1 - 2\alpha = m$, $\quad \alpha_4 = -\alpha = \frac{m-1}{2}$

Hence, $\quad \eta = \frac{y}{x^{\frac{1-m}{2}}} \qquad \hat{f} = \frac{u}{x^m} \qquad \hat{g} = \frac{v}{x^{\frac{m-1}{2}}} \qquad \hat{u}_e = \frac{u_e}{x^m}$

<u>First transform $(x\ y) \Rightarrow (\xi\ \eta)$</u>:

$$u = \frac{\partial \psi}{\partial y} \quad \Rightarrow \quad u = \frac{1}{\Delta}\frac{\partial \psi}{\partial \eta}$$

$$\frac{\tau}{\rho} = \nu \frac{\partial u}{\partial y} \quad \Rightarrow \quad \frac{\tau}{\rho} = \frac{\nu}{\Delta}\frac{\partial u}{\partial \eta}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial u}{\partial y} = u_e \frac{du_e}{dx} + \frac{\partial}{\partial y}\left(\frac{\tau}{\rho}\right) \quad \Rightarrow \quad \left(\frac{1}{\Delta}\frac{\partial \psi}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi} - \cancel{\frac{\eta}{\Delta}\frac{d\Delta}{d\xi}\frac{\partial u}{\partial \eta}}\right) - \left(\frac{\partial \psi}{\partial \xi} - \cancel{\frac{\eta}{\Delta}\frac{d\Delta}{d\xi}\frac{\partial \psi}{\partial \eta}}\right)\left(\frac{1}{\Delta}\frac{\partial u}{\partial \eta}\right) = u_e\frac{du_e}{d\xi} + \frac{1}{\Delta}\frac{\partial}{\partial \eta}\left(\frac{\tau}{\rho}\right)$$

$$\text{or} \quad \frac{\partial \psi}{\partial \eta}\frac{\partial u}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial u}{\partial \eta} = u_e\frac{du_e}{d\xi}\Delta + \frac{\partial}{\partial \eta}\left(\frac{\tau}{\rho}\right)$$

<u>Now substitute $\left(\psi\ u\ \frac{\tau}{\rho}\right)$ in terms of $(F\ U\ S)$</u>: note: $u_e^+(\xi)$ only, $n^+(\xi)$ only, etc.

$$u = \frac{1}{\Delta}\frac{\partial \psi}{\partial \eta} \quad \Rightarrow \quad u_e^- + U(u_e^+ - u_e^-) = \frac{1}{\Delta}\left[n^- + \frac{\partial F}{\partial \eta}(n^+ - n^-)\right] \quad \Rightarrow \quad \boxed{U = \frac{\partial F}{\partial \eta}}$$

$$\frac{\tau}{\rho} = \frac{\nu}{\Delta}\frac{\partial u}{\partial \eta} \quad \Rightarrow \quad \frac{\Delta}{\xi}(u_e^+ - u_e^-)^2 S = \frac{\nu}{\Delta}(u_e^+ - u_e^-)\frac{\partial U}{\partial \eta} \quad \Rightarrow \quad \boxed{S = \frac{\nu\xi(u_e^+ - u_e^-)}{(n^+ - n^-)^2}\frac{\partial U}{\partial \eta}}$$

$$\frac{\partial \psi}{\partial \eta}\frac{\partial U}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial U}{\partial \eta} = u_e\frac{du_e}{d\xi}\Delta + \frac{\partial}{\partial \eta}\left(\frac{\tau}{\rho}\right) \quad \Rightarrow \quad \left[n^- + \frac{\partial F}{\partial \eta}(n^+ - n^-)\right]\left[\frac{du_e^-}{d\xi} + U\frac{d}{d\xi}(u_e^+ - u_e^-) + \frac{\partial U}{\partial \xi}(u_e^+ - u_e^-)\right]$$

$$- \left[\frac{dn^-}{d\xi} + F\frac{d}{d\xi}(n^+ - n^-) + \frac{\partial F}{\partial \xi}(n^+ - n^-)\right]\left[\frac{\partial U}{\partial \eta}(u_e^+ - u_e^-)\right] = u_e^-\frac{du_e^-}{d\xi}\Delta + \frac{\Delta}{\xi}(u_e^+ - u_e^-)^2\frac{\partial S}{\partial \eta}$$

Mult. through by $\frac{\xi}{(u_e^+ - u_e^-)(n^+ - n^-)}$:

$$\cancel{\frac{n^-}{n^+ - n^-}\frac{\xi}{u_e^+ - u_e^-}\frac{du_e^-}{d\xi}} + \beta_u\frac{\partial F}{\partial \eta}U + \xi\frac{\partial F}{\partial \eta}\frac{\partial U}{\partial \xi} - \frac{\xi}{n^+ - n^-}\frac{dn^-}{d\xi} - \beta_n F\frac{\partial U}{\partial \eta} - \xi\frac{\partial F}{\partial \xi}\frac{\partial U}{\partial \eta}$$

$$= \cancel{\frac{u_e^-\Delta}{n^+ - n^-}\frac{\xi}{u_e^+ - u_e^-}\frac{du_e^-}{d\xi}} + \frac{\partial S}{\partial \eta}$$

Finally: $\Rightarrow$

$$\boxed{\frac{\partial S}{\partial \eta} + \beta_n F\frac{\partial U}{\partial \eta} - \beta_u\frac{\partial F}{\partial \eta}U - \frac{\xi}{n^+ - n^-}\frac{dn^-}{d\xi} = \xi\left(\frac{\partial F}{\partial \eta}\frac{\partial U}{\partial \xi} - \frac{\partial F}{\partial \xi}\frac{\partial U}{\partial \eta}\right)}$$

where:

$$\beta_u \equiv \frac{\xi}{u_e^+ - u_e^-}\frac{d}{d\xi}(u_e^+ - u_e^-)$$

$$\beta_n \equiv \frac{\xi}{n^+ - n^-}\frac{d}{d\xi}(n^+ - n^-)$$

<u>For similarity</u> $\left(\frac{\partial}{\partial \xi} = 0\right)$, we must have:

- $\beta_u, \beta_n = \text{constants} \rightarrow (u_e^+ - u_e^-) = C_1\xi^{\beta_u} \qquad (n^+ - n^-) = C_2\xi^{\beta_n}$
- $\frac{\nu\xi(u_e^+ - u_e^-)}{(n^+ - n^-)^2} \sim \xi^{1+\beta_u-2\beta_n} = \text{constant} \rightarrow \beta_n = \frac{1+\beta_u}{2}$
- $\frac{\xi}{n^+ - n^-}\frac{dn^-}{d\xi} \sim \xi^{1-\beta_n}\frac{dn^-}{d\xi} = \text{constant} \rightarrow \frac{dn^-}{d\xi} \sim \xi^{\beta_n - 1} \rightarrow n^- = C_3\xi^{\beta_n}$

The last constraint implies that $n^+ - n^- = n^+ + C_3\xi^{\beta_n} = C_2\xi^{\beta_n} \rightarrow n^+ = (C_2 - C_3)\xi^{\beta_n}$
i.e. n^+ and n^- must independently have the same power-law exponent, and hence $u_e^+ \equiv n^+/\Delta$ and $u_e^- \equiv n^-/\Delta$ must likewise.

But we also require that $u_e^-\frac{du_e^-}{d\xi} = u_e^+\frac{du_e^+}{d\xi} \rightarrow u_e^{+2} = u_e^{-2} + \text{const} \rightarrow u_e^{+2} - u_e^{-2} = (u_e^+ - u_e^-)(u_e^+ + u_e^-) = \text{const.}$

so $\xi^{\beta_u} \sim (u_e^+ + u_e^-) \sim (u_e^+ - u_e^-)^{-1} \sim \xi^{-\beta_u}$. <u>Only possible if $\beta_u = 0$</u> i.e. <u>u_e^+ and u_e^- both constant</u>