Lecture 30 Nov 21, 03

Compressible TSL

8.17 A) Defn. and implications of compressibility

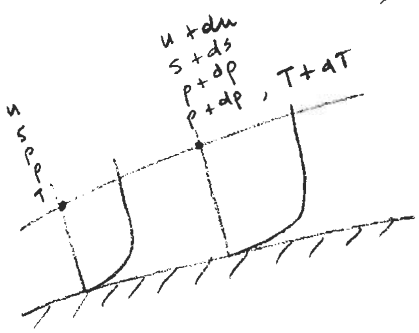
B) Special Soln

C) Reynolds Analogy

Reading: Sch 327 – 330, 340 – 352

Wh 184 - 200, 576 – 616.

A)



· Compressibility implies the effect of M on BL can not be ignored ($\rho \neq$ const)

· Using the ideal gas law

$$\rho = P/RT$$

$$\ln\rho = \ln P - \ln T - \ln R$$

$$\Rightarrow \frac{d\rho}{\rho} = \frac{dp}{p} - \frac{dT}{T}$$

In the outer potential flow (isentropic + adiabatic)

$$ds = 0 = c_p \ln T - \ln P = 0$$

$$\frac{(\gamma-1)}{\gamma}\frac{dP}{P} = \frac{dT}{T}$$

$$\therefore \frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dp}{p} = \frac{1}{\gamma-1}\frac{dT}{T}$$

The momentum equation along a streamline

$$dp + \rho u\, du = 0$$

$$\therefore \quad \frac{dp}{p} = -\frac{\rho u\, du}{p}$$

$$= -\gamma M^2 \frac{du}{u}$$

$$\therefore \quad \frac{d\rho}{\rho} = -M^2 \frac{du}{u}$$

For a thin shear layer

$$\frac{dp_e}{p_e} = -M_e^2 \frac{du_e}{u_e}$$

Therefore the criterion is that if $M_e^2 << 1$ then $p_e = \text{const.}$

Inside the shear layer $dS \neq 0$, so the energy eqn must be introduced. The TSL equations are

$$\vec{\nabla}\cdot(\rho\vec{u}) = 0 = \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial p}{\partial y} \approx 0$$

energy $\longrightarrow \rho\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = u\frac{\partial p_e}{\partial x} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2$

where, $\quad h = c_p T \qquad c_p = \frac{\gamma R}{\gamma - 1}$

$\quad = e + p/\rho$

Note: $\quad \mu = \mu(T), \quad k = k(T)$ (thermal conductivity)

Relative Scales.

$$\underbrace{\rho\left(u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}\right)}_{O(1)} = \underbrace{u\frac{dp_e}{dx}}_{O(M_e^2)} + \underbrace{\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)}_{O(1)} + \underbrace{\mu\left(\frac{\partial u}{\partial y}\right)^2}_{O(M_e^2)}$$

can be present in low speed flows.

~~Also express as~~

~~Define non-dimensional Pr number.~~

$$\cancel{\rho\left(u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}+\mu u\frac{\partial u}{\partial y}\right)}$$

~~What~~, $H_0 = h + u^2/2$ — stagnation enthalpy. $C_pT_0 = H_0$

Define non-dimension Pr #

$$Pr \equiv \frac{\mu C_p}{k}, \quad h = C_pT, \quad C_p = \frac{\gamma R}{\gamma-1}$$

We can rewrite energy equation as. (* + u · X-mom)

$$\rho u\left[\frac{\partial h}{\partial x}+u\frac{\partial u}{\partial x}\right]+\rho v\left[\frac{\partial h}{\partial y}+u\frac{\partial u}{\partial y}\right] = \frac{1}{Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial h_0}{\partial y}\right)+\frac{\partial}{\partial y}\left((1-1/Pr)\mu u\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \rho\left[u\frac{\partial h_0}{\partial x}+v\frac{\partial h_0}{\partial y}\right] = \frac{1}{Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial h_0}{\partial y}\right)+\frac{\partial}{\partial y}\left[(1-1/Pr)\mu u\frac{\partial u}{\partial y}\right]$$

=> Special Solution

If $Pr = 1$, the above eqn simplifies to

$$LHS = \frac{\partial}{\partial y}\left(\mu\frac{\partial h_0}{\partial y}\right)$$

The equation admits a special solution:

$$h_0(y) = \text{const.}$$

Since $h_0 = h + \frac{1}{2}u^2$

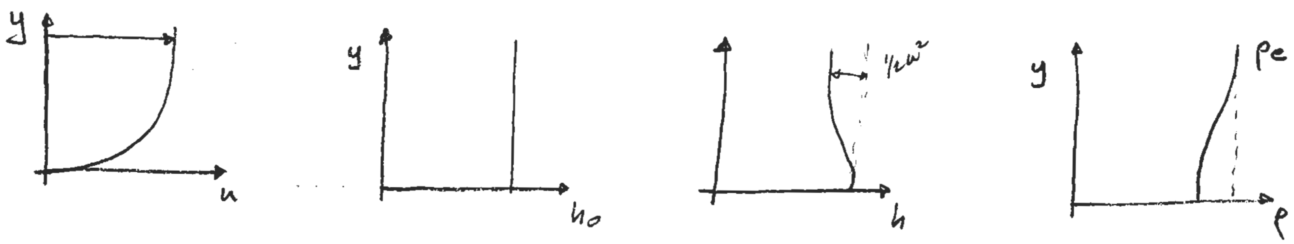
$$\frac{\partial h_0}{\partial y} = \frac{\partial h}{\partial y} + u\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \frac{\partial h}{\partial y} = \frac{\partial h_0}{\partial y} = 0 \quad \text{at the wall}$$

no temp gradient — adiabatic

Thus, this represents zero ht transfer at the wall. $Pr = 1$ implies perfect balance between viscous dissipation and ht conduction so that $h_0 =$ const in BL.

Note: · $Pr \approx 1$ is a good approximation for gases.
· pressure gradient does not show up.



Since $p \approx p_e$

$$\frac{d\rho}{\rho} = -\frac{dT}{T} = -\frac{dh}{h}$$

$$\therefore \left(\frac{\rho}{\rho_e}\right) = \left(\frac{h_e}{h}\right)$$

A second special case with $P_r = 1$ is $\frac{dp_e}{dx} = 0$. Comparing

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \quad — (1)$$

same governing eqn

$$u\frac{\partial h_0}{\partial x} + v\frac{\partial h_0}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial h_0}{\partial y}\right) \quad — (2)$$

Admits solution $h = h(u)$ (or $h_0(y) = A\,u(y) + B$ satisfies (2))

A, B set by B.C at $y=0$, $y = y_e$

$$\therefore \quad \frac{\partial h}{\partial y} = \frac{dh}{du}\frac{\partial u}{\partial y}$$

Substituting into energy eqn gives.

$$\frac{dh}{du}\underbrace{\left[\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right]}_{0 \text{ (mom eqn)}} = \left(1 + \frac{d^2 h}{du^2}\right)\mu\left(\frac{\partial u}{\partial y}\right)^2$$

$$\therefore \quad \frac{d^2 h}{du^2} = -1 \quad \text{or} \quad \begin{aligned} H_0 &= Au + B. \\ h &= Au + B - u^2/2 \end{aligned}$$

$$\therefore \quad H_0 = h + u^2/2 = h_w + (H_e - h_w)\, u/u_e$$

$$\frac{1}{P_r}\frac{\partial h_0}{\partial y}\Big|_{wall} = q_w \neq 0$$

(can have ht transfer at wall)