Thin Wing Theory

Objective:

- Derive the equations of motion governing the subsonic flow around thin wings
- Use matched asymptotic expansions
  - Physical insight increase
  - Contrast airfoil with slender body of revolution

Assumptions:

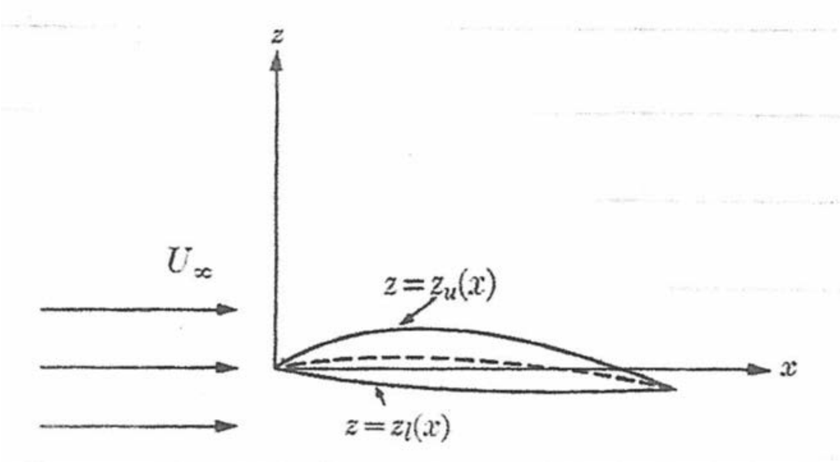
- Steady flow
- Inviscid flow
- Irrotational flow
- Adiabatic fluid
- Ideal gas
- Constant specific heats
- Wing is in the x,y-plane

Notation:

τ = Thickness ratio
α = Angle of attack
θ = A measure of the amount of camber
$\bar{g}(x)$ = Thickness distribution along the chord
$\bar{h}(x)$ = Camber distribution along the chord
$\bar{g}$ and $\bar{h}$ are both smooth
$\bar{g}'$ and $\bar{h}'$ are of order of unity everywhere
$\tau << 1, \theta << 1, \alpha << 1$

$$z_u = \epsilon \bar{f}_u(x) = \tau \bar{g}_u(x) + \theta \bar{h}_u(x) - \alpha x$$

$$z_l = \epsilon \bar{f}_l(x) = -\tau \bar{g}_u(x) + \theta \bar{h}_u(x) - \alpha x$$



ϵ = Small dimensionless quantity measuring maximum cross wise extension of the airfoil

We will treat two dimensional flow. Extension to three dimensions is straightforward. We week the leading terms in a series expansion in ϵ of Φ. We have shown the governing equations to be:

$$(a^2 - \Phi_x^2)\Phi_{xx} + (a^2 - \Phi_z^2)\Phi_{zz} - 2\Phi_x\Phi_z\Phi_{xz} = 0$$

$$a^2 = a_\infty^2 - \frac{\gamma - 1}{2}\left(\Phi_x^2 + \Phi_z^2 - U_\infty^2\right)$$

$$C_p = \frac{2}{\gamma M_\infty^2}\left[\left[1 - \frac{\gamma-1}{2a_\infty^2}\left(\Phi_x^2 + \Phi_z^2 - U_\infty^2\right)\right]^{\gamma/(\gamma-1)} - 1\right]$$

The boundary conditions are:

A) $\bar{Q} = U_\infty \overrightarrow{i}, (x,z) \to \infty$

B) Flow is tangent to the airfoil surface

$$\Phi_z/\Phi_x = \epsilon\frac{d\bar{f}_u}{dx}, z = \epsilon\bar{f}_u$$

$$\Phi_z/\Phi_x = \epsilon\frac{d\bar{f}_l}{dx}, z = \epsilon\bar{f}_l$$

C) Pressure is continuous at the trailing edge - Kutt-Jovkowsky condition

Assume in outer expansion:

$$\Phi^0 = U_\infty\left[\Phi^0(x,z) + \epsilon\Phi_1^0(x,z) + ...\right]$$

Since the airflow in the limit of $\epsilon \to 0$ collapses to a line parallel to the free stream, the zeroth order term must represent parallel undisturbed flow. Thus,

$$\Phi_0^0 = x$$

Substituting, we obtain:

$$(1-M_\infty^2)\Phi^0_{i_{xx}} + \Phi^0_{i_{zz}} = 0$$

$$\Phi^0_{i_x} + \Phi^0_{i_z} \to 0, \text{for} \sqrt{x^2+z^2}$$

The remaning boundary conditions belong to the inner region and are to be obtained by matching. We assume:

$$\Phi^i = U_\infty\left[\Phi^i_0(x,\bar{z}) + \epsilon\Phi^i_1(x,\bar{z}) + \epsilon^2\Phi^i_2(x,\bar{z}) + \ldots\right]$$

$$\bar{z} = z/\epsilon$$

The stretching enables us to focus on the flow in the immediate neighborhood of the airfoil in the limit of $\epsilon \to 0$ since the airfoil shape then remains independent of ϵ and the width of the inner region becomes of order unity.

The zeroth-order inner term is that of a parallel flow, that is:

$$\Phi^i_0 = x$$

(Both inner and outer flow must be parallel in the limit $\epsilon \to 0$.)

Examining the W-velocity component in the inner region, we may write:

$$W = \Phi^i_z = \frac{1}{\epsilon}\Phi^i_{\bar{z}} = U_\infty\left[\Phi^i_{1_{\bar{z}}}(x,\bar{z}) + \epsilon\Phi^i_{2_{\bar{z}}}(x,\bar{z})\right]$$

Since Q must vanish in the limit of zero ϵ, Φ^i_1 must be independent of $\bar{z}$. Hence, let:

$$\Phi^i_1 = \bar{g_1}(x)$$

$\bar{g_1}(x)$ may be different above and below the airfoil

We now write:

$$\Phi^i = U_\infty\left[x + \epsilon\bar{g_1}(x) + \epsilon^2\Phi^i_2(x,\bar{z}) + \ldots\right]$$

Substituting the above equation into the governing equation and boundary conditions, we obtain:

$$\Phi^i_{2_{\bar{z}\bar{z}}} = 0$$

$$\Phi^i_{2_{\bar{z}}} = \frac{d\bar{f}_u}{dx} \text{ for } \bar{z} = \bar{f}_u(x)$$

$$\Phi^i_{2_{\bar{z}}} = \frac{d\bar{f}_l}{dx} \text{ for } \bar{z} = \bar{f}_l(x)$$

The the solution must be linear in $\bar{z}$,

$$\Phi^i_2 = \bar{z}\frac{d\bar{f}_u}{dx} + \bar{g_{2_u}}(x)\ ,\ \bar{z} \geq \bar{f}_u$$

$$\Phi^i_2 = \bar{z}\frac{d\bar{f}_l}{dx} + \bar{g_{2_l}}(x)\ ,\ \bar{z} \geq \bar{f}_l$$

The inner solution cannot give vanishing disturbances at infinity since the boundary condition belongs to the outer region. We need to match the inner and outer solutions.

We will employ the asymptotic matching principle to complete the required matching.

From Φ^i above we note that W^i is independent of $\bar{z}$ to lowest order. Thus, in the outer limit at $\bar{z} = \infty$, we have:

$$W^{i0} = \frac{1}{\epsilon}\Phi^i_{\bar{z}}(x,\infty) = \epsilon\frac{d\bar{f}_u}{dx}$$

and

$$W^0 = \epsilon\Phi^0_{1z}$$

Equating the inner limit $(z = 0^+)$, we obtain:

$$\Phi^0_{1z}(x,0^+) = \frac{d\bar{f}_u}{dx}$$

$$\Phi^0_{1z}(x,0^-) = \frac{d\bar{f}_l}{dx}$$

Matching the potential, we find:

$$\bar{g_{1_u}}(x) = \Phi^0_1(x,0^+)$$

To determine $\bar{g_2}$ it is necessary to include higher order terms in the outer solution.

First, we write the two-term outer flow in inner variables,

$$\Phi^0 = U_\infty\big(x + \epsilon\Phi^0_1(x,\epsilon\bar{z}) + \ldots\big)$$

Taking the three term inner expansion of this:

$$\Phi^0 = U_\infty\big(x + \epsilon\Phi^0_1(x,0^+) + \epsilon^2\bar{z}\Phi^0_{1z}(x,0^+) + \ldots\big)$$

Rewriting the above equation in outer variables:

$$\Phi^0 = U_\infty\big(x + \epsilon\Phi^0_1(x,0^+) + \epsilon z\Phi^0_{1z}(x,0^+) + \ldots\big)$$

The three-term inner expansion, expressed in outer variables, reads:

$$\Phi^i = U_\infty\big(x + \epsilon\bar{g_{1_u}}(x) + \epsilon z\frac{d\bar{f}_u}{dx} + \epsilon^2\bar{g_{2_u}}(x) + \ldots\big)$$

Now equate the two-term outer expansion of the three term inner expansion:

$$\Phi^i = U_\infty\big(x + \epsilon\bar{g_{1_u}}(x) + \epsilon z\frac{d\bar{f}_u}{dx} + \ldots\big)$$

To the three term inner expansion of the two-term outer expansion leads to:

$$\bar{g_{1_u}} = \Phi^0_i(x,0^+)$$

$$\Phi^0_{1_z}(x,0^+) = \frac{d\bar{f}_u}{dx}$$

And C_p takes the following form:

$$C_p = -2\epsilon\Phi^0_{1x}(x,0^\pm)$$

"+" → upper surface
"-" → lower surface