Viscous Flow: Stress Strain Relationship

Objective: Discuss assumptions which lead to the stress-strain relationship for a Newtonian, linear viscous fluid:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\frac{\partial u_k}{\partial x_k}$$

$$\frac{\partial u_k}{\partial x_k} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \bullet \vec{V}$$

where $\mu =$ dynamic viscosity coefficient
λ = bulk viscosity coefficient

Note: $\delta_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \text{shear strain rate in } x_i, x_j \text{ plane}$$

Thus, written in terms of the strain rates, the stress tensor is:

$$\tau_{ij} = \underbrace{2\mu\varepsilon_{ij}}_{\substack{\text{viscous stress} \\ \text{due to shearing} \\ \text{of a fluid element}}} + \underbrace{\delta_{ij}\lambda\underbrace{\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)}_{\substack{\text{using indicial notation} \\ \text{this is } \varepsilon_{kk}}}}_{\substack{\text{viscous stress due to an} \\ \text{overall compression or} \\ \text{expansion of the fluid} \\ \text{element's volume}}}$$

This stress-strain relationship can be derived by the following two assumptions:

1. The shear stress is independent of a rotation of the coordinate system
2. The shear stress is at most a linear function of the strain rate tensor. So, for example, τ_{xy}:

$$\tau_{xy} = a_{11}\varepsilon_{xx} + a_{12}\varepsilon_{xy} + a_{13}\varepsilon_{xz} + a_{22}\varepsilon_{yy} + a_{23}\varepsilon_{yz} + a_{33}\varepsilon_{zz}$$

Clearly, assumption #2 gives 6 unknowns per shear, a_{11}, a_{12}, etc. Note: why do $\varepsilon_{yx}, \varepsilon_{zx}$ & ε_{zy} not appear in this expression for τ_{xy}? The total number of unknowns for all stresses are: 36. But, this can be eventually reduced by applying #1 to the most general linear form to the two unknowns μ & λ.

Stokes' hypothesis

Stokes hypothesized that the total normal viscous stress on a fluid element surface,

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = (2\mu + 3\lambda)(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$$

should be zero, i.e. $\tau_{xx} + \tau_{yy} + \tau_{zz} = 0$ so that the normal force (stress) on a surface is only that due to pressure. This requires that

$$2\mu + 3\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}\mu$$

Comments

- Experimental studies have indicated that $\lambda \neq -\frac{2}{3}\mu$ and in general is not negative!
- For incompressible flow, $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \nabla \cdot \vec{V} = 0$ so $\tau_{xx} + \tau_{yy} + \tau_{zz} = 0$ regardless of λ.
- For most compressible flows $\nabla \cdot \vec{V}$ is small compared to shearing strains (i.e. $\varepsilon_{xy}, \varepsilon_{yz}$, etc.) so again, Stokes' hypothesis has little impact. So, as a result, common practice is to assume that $\lambda = -\frac{2}{3}\mu$.