Quick Visit to Bernoulli Land

Although we have seen the Bernoulli equation and seen it derived before, this next note shows its derivation for an uncompressible & inviscid flow. The derivation follows that of Kuethe &Chow most closely (I like it better than Anderson).[1]

Start from inviscid, incompressible momentum equation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla p$$

There is a vector calculus identity:

$$\vec{u} \bullet \nabla \vec{u} = \nabla \left(\frac{1}{2} |\vec{u}|^2 \right) - \vec{u} \times \underbrace{(\nabla \times \vec{u})}_{\vec{\omega}, vorticity}$$

$$\Rightarrow \boxed{\frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{1}{2} |\vec{u}|^2 \right) + \frac{1}{\rho} \nabla p = \vec{u} \times \vec{\omega}}$$

From here, we can make the final re-arrangement:

$$\boxed{\nabla \left(p + \frac{1}{2} \rho |\vec{u}|^2 \right) = \rho \vec{u} \times \vec{\omega} - \rho \frac{\partial \vec{u}}{\partial t}}$$

Two common applications:

1. <u>Steady irrotational flow</u>

$$\underbrace{\frac{\partial \vec{u}}{\partial t} = 0}_{Steady} \qquad \underbrace{\vec{\omega} = 0}_{Irrotational}$$

$$\Rightarrow \nabla \left(p + \frac{1}{2} \rho |\vec{u}|^2 \right) = 0$$

$$\Rightarrow \boxed{p + \frac{1}{2} \rho |\vec{u}|^2 = const.\ for\ entire\ flow}$$

[1] Kuethe and Chow, 5th Ed. Sec 3.3-3.5

2. Steady but rotational flow

$$\underbrace{\frac{\partial \vec{u}}{\partial t} = 0}_{Steady} \qquad \underbrace{\vec{\omega} \neq 0}_{Rotational}$$

$$\Rightarrow \nabla\left(p + \frac{1}{2}\rho|\vec{u}|^2\right) = \rho\vec{u} \times \vec{\omega}$$

This is a vector equation. If we dot product this into the streamwise direction:

$$\vec{s} \equiv \frac{\vec{u}}{|\vec{u}|} \leftarrow \text{streamwise direction}$$

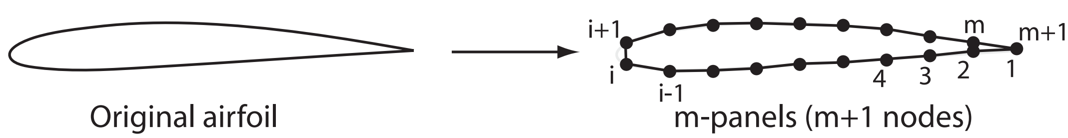
$$\Rightarrow \vec{s} \bullet \nabla\left(p + \frac{1}{2}\rho|\vec{u}|^2\right) = \rho \underbrace{\vec{s} \bullet (\vec{u} \times \vec{\omega})}_{=0,\ (\vec{u}\times\vec{\omega})\perp\vec{u}}$$

$$\Rightarrow \frac{d}{ds}\left(p + \frac{1}{2}\rho|\vec{u}|^2\right) = 0$$

$$\Rightarrow \boxed{p + \frac{1}{2}\rho|\vec{u}|^2 = const.\, along\ streamline}$$

Vortex Panel Methods[2]

Step#1: Replace airfoil surface with panels



Step #2: Distribute singularities on each panel with unknown strengths

In our case we will use vortices distributed such that their strength varies linearly from node to node:

Recall a point vortex at the origin is:

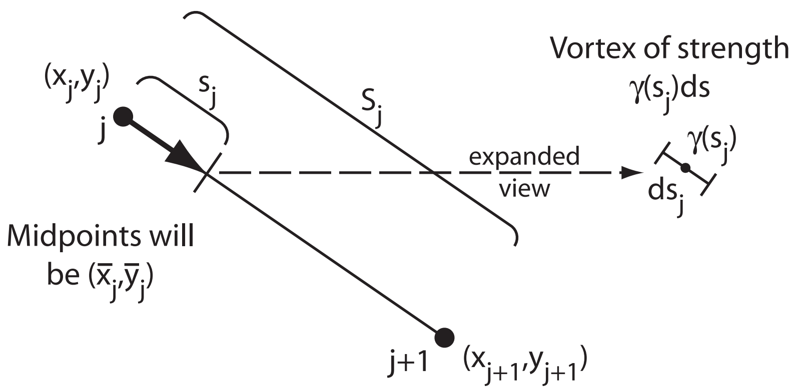
$$\phi = -\frac{\Gamma}{2\pi}\theta = -\frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

[2] Kuethe and Chow, 5th Ed. Sec. 5.10

A point vortex at $\hat{x}, \hat{y}$ is:

$$\phi = -\frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y-\hat{y}}{x-\hat{x}}\right)$$

Next, consider an arbitrary panel:



At any s_j, we will place a vortex with strength $\gamma\left(s_j\right)ds$:

$$\Rightarrow d\phi\left(x,y\right) = -\frac{\gamma(s_j)ds}{2\pi}\tan^{-1}\left(\frac{y-\hat{y}_j}{x-\hat{x}_j}\right)$$

where

$$\hat{x}_j \equiv x_j + (x_{j+1}-x_j)\frac{s_j}{S_j}$$

$$\hat{y}_j \equiv y_j + \left(y_{j+1}-y_j\right)\frac{s_j}{S_j}$$

Thus, the potential at any (x,y) due to the entire panel j is:

$$\boxed{\phi_j\left(x,y\right) \equiv -\int_0^{S_j}\frac{\gamma\left(s_j\right)}{2\pi}\tan^{-1}\left(\frac{y-\hat{y}_j}{x-\hat{x}_j}\right)ds}$$

We will assume linear varying γ on each panel:

$$\boxed{\gamma\left(s_j\right) = \gamma_j + \left(\gamma_{j+1}-\gamma_j\right)\frac{s_j}{S_j}}$$

With this type of panel, we have m+1 unknowns = $\gamma_{1,}\gamma_{2,}\gamma_{3...}\gamma_{m-1,}\gamma_{m,}\gamma_{m+1}$, so we need m+1 equations.

<u>Step#3: Enforce Flow Tangency at Panel Midpoints</u>

The next step is to enforce some approximation of the boundary conditions at the airfoil surface. To do this, we will enforce flow tangency at the midpoint of each panel.

Panel method lingo: *control point* is a location where $\vec{u} \bullet \vec{n} = o$ is enforced.

To do this, we need to find the potential and the velocity at each control point.

The potential has the following form:

$$\phi = \begin{pmatrix} freestream \\ potential \end{pmatrix} + \sum_{\#\,panels} \begin{pmatrix} individual\ panel \\ potential \end{pmatrix}$$

Suppose freestream has angle α:

$$\boxed{\phi(x,y) = V_\infty \left(x\cos\alpha + y\sin\alpha\right) - \sum_{j=1}^{m} \int_0^{S_j} \frac{\gamma(s_j)}{2\pi} \tan^{-1}\left(\frac{y-\hat{y}_j}{x-\hat{x}_j}\right) ds}$$

The required boundary condition is $\frac{\partial \phi}{\partial n_i}\left(\bar{x}_{i,}\bar{y}_i\right) = 0 \;\; for\,all\;\; i = 1 \to m$

So, let's carry this out a little further:

$$\frac{\partial \phi}{\partial n_j}\left(\bar{x}_{i,}\bar{y}_i\right) = \underbrace{V_\infty\left(\cos\alpha\vec{i} + \sin\alpha\vec{j}\right)\bullet\vec{n}_i}_{\text{component of freestream normal to surface of panel i}} - \underbrace{\sum_{j=0}^{m}\int_0^{S_j} \frac{\gamma(s_j)}{2\pi}\left\{\frac{\partial}{\partial n_i}\left[\tan^{-1}\left(\frac{y-\hat{y}_j}{x-\hat{x}_j}\right)\right]_{\bar{x}_i,\bar{y}_i}\right\} ds}_{\text{normal velocity due to panel j at control point of panel i}} = 0$$

And recall $\gamma\left(s_j\right) = \gamma_j + \left(\gamma_{j+1} - \gamma_j\right)\frac{s_j}{S_j}$.

We can re-write these integrals in a compact notation:

$$\int_0^{S_j} \frac{\gamma(s_j)}{2\pi}\left\{\frac{\partial}{\partial n_i}\left[\tan^{-1}\left(\frac{y-\hat{y}_j}{x-\hat{x}_j}\right)\right]_{\bar{x}_i,\bar{y}_i}\right\} ds = \underbrace{C_{n1_{ij}}\gamma_j + C_{n2_{ij}}\gamma_{j+1}}_{C_{n1_{ij}} = \text{Influence of panel j due to node j on control point of panel i}}$$

i.e. $C_{n1_{ij}}\gamma_j =$ normal velocity from panel j due to node j on control point of panel i .

$C_{n2_{ij}} =$ Influence of panel j due to node $j+1$ on control point at panel i

$\Rightarrow$ Total normal velocity at control point of panel i due to panel j $= C_{n1_{ij}}\gamma_j + C_{n2_{ij}}\gamma_{j+1}$

So, let's look at the control point normal velocity

So, for panel i, flow tangency looks like:

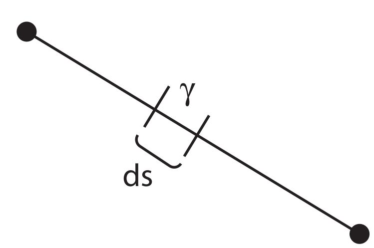
$$\sum_{j=1}^{m}\left(C_{n1_{ij}}\gamma_j + C_{n2_{ij}}\gamma_{j+1}\right) = \underbrace{V_\infty\left(\cos\alpha\vec{i} + \sin\alpha\vec{j}\right)\bullet\vec{n}_i}_{V_\infty n_i}, \text{ for all } i = 1 \to m$$

We can write this as a set of m equations for m+1 unknowns.

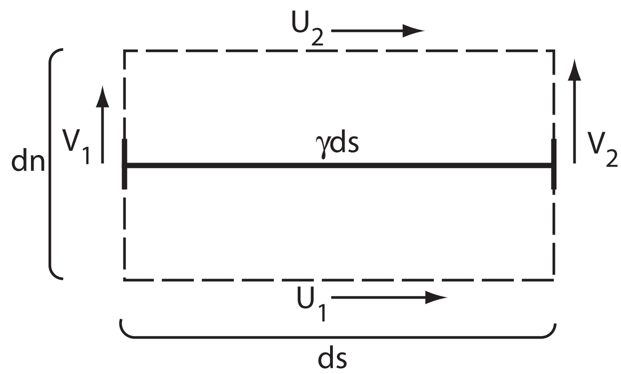
Question: What can we do for one more equation?

<u>Step#4: Apply Kutta condition</u>

We need to relate Kutta condition to the unknown vortex strengths γ_j. To do this, consider a portion of a vortex panel.



Put a contour about differential element ds



$$\Gamma = -\oint \vec{u} \cdot d\vec{s}$$

Recall: $\Rightarrow \gamma ds = -\left[V_2 dn - U_2 ds - V_1 dn + U_1 ds\right]$

$$= \left(V_1 - V_2\right) dn - \left(U_1 - U_2\right) ds$$

Now let $dn \,\&\, ds \to 0$:

$dn \to O,\ \gamma ds = -\left(U_1 - U_2\right) ds$

$\gamma = U_2 - U_1,\ \text{or}$

$\gamma = U_{top} - U_{bottom},\ \text{in general}$

So, since the Kutta condition requires $U_{top} = U_{bottom}$ at TE:

$$\boxed{\gamma_{t.e.} = 0,\ \text{Kutta condition}}$$

For the vortex panel method, this means:

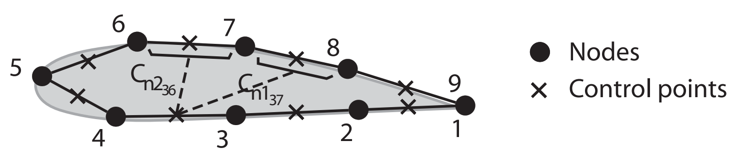
$$\boxed{\gamma_1 + \gamma_{m+1} = 0}$$

<u>Step#5: Set-up System of Equations & Solve</u>

$$
\begin{array}{r}
\vec{u}\bullet\vec{n}=0@i=1 \\
\vec{u}\bullet\vec{n}=0@i=2 \\
\\ \\ \\ \\ \\
\vec{u}\bullet\vec{n}=0@i=m \\
Kutta
\end{array}
\quad
\underbrace{\begin{bmatrix}
I_{11} & I_{12} & & & I_{19} \\
I_{21} & I_{22} & & & \\
I_{31} & I_{32} & I_{33} & & \\
I_{41} & & & & \\
I_{51} & & & & \\
I_{61} & & & & \\
I_{71} & & & & \\
I_{81} & & & & \\
1 & 0 & 0 & 0 & 1
\end{bmatrix}}_{I}
\underbrace{\begin{bmatrix}
\gamma_1 \\ \gamma_2 \\ \gamma_3 \\ . \\ . \\ . \\ \\ \gamma_8 \\ \gamma_9
\end{bmatrix}}_{\gamma}
=-
\underbrace{\begin{bmatrix}
V_{\infty_{n1}} \\ V_{\infty_{n2}} \\ . \\ . \\ . \\ \\ \\ V_{\infty_{n8}} \\ 0
\end{bmatrix}}_{V_{\infty_n}}
$$

Where I_{ij} = total influence of node j at control point i

For example: $I_{37} = C_{n1_{37}} + C_{n2_{36}}$



The problem thus reduces to $\mathbf{Ax} = \mathbf{b}$, or, using our notation

$\mathbf{I\gamma} = \mathbf{V}_{\infty_n}$,

which we solve to find the vector of γ's!

Step #6: Post-processing

The final step is to post-process the results to find the pressures and the lift acting on the airfoil.

$$L' = \rho V_\infty \Gamma = \rho V_\infty \oint_{airfoil} \gamma ds$$

So, for our method, this reduces to:

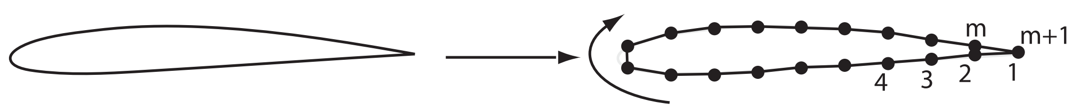
$$L' = \frac{1}{2}\rho V_\infty \sum_{i=1}^{m} (\gamma_i + \gamma_{i+1}) S_i$$

$$\Rightarrow \boxed{C_l = \frac{L'}{\frac{1}{2}\rho V_\infty^2 c} = \sum_{i=1}^{m} \left(\frac{\gamma_i}{V_\infty} + \frac{\gamma_{i+1}}{V_\infty} \right) \frac{S_i}{c}}$$

Vortex Panel Method Summary

In practice, the vortex panel method used for airfoil flows is a little different than the strategy used in the windy city problem. Here's a summary:

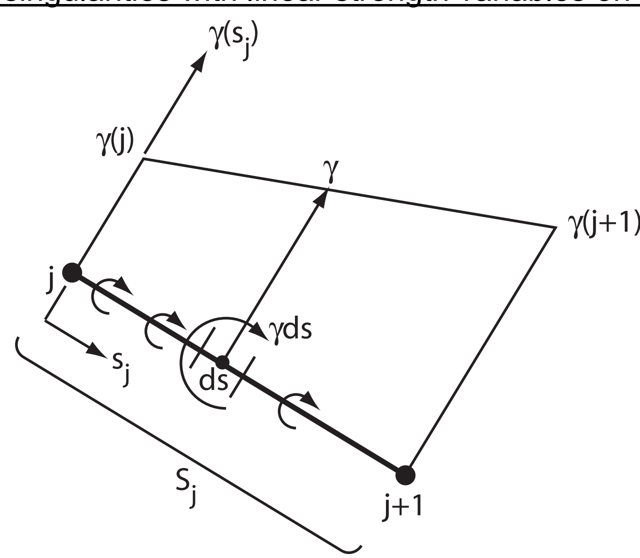
Step #1: Replace airfoil surface with panels



Note: the trailing edge is double-numbered $\Rightarrow m+1$ points, m panels.

Step #2: Distribute vortex singularities with linear strength variables on each panel

$$\boxed{\gamma(s_j) = \gamma_j + (\gamma_{j+1} - \gamma_j)\frac{s_j}{S_j}}$$



This means we have m+1 unknowns:

$\gamma_1, \gamma_2, \gamma_3, \ldots\ldots\ldots \gamma_m, \gamma_{m+1}$

Step #3: Enforce flow tangency at panel midpoints

$\vec{u} \bullet \vec{n} = 0$ at midpoint of every panel

$\Rightarrow \boxed{m \text{ equations}}$

Step#4: Apply Kutta condition

Kutta condition becomes:

$\gamma_{t.e.} = 0 \Rightarrow \boxed{\gamma_1 + \gamma_{m+1} = 0}$

$\Rightarrow \boxed{m+1 \text{ equations } \& \ m+1 \text{ unknowns}}$