Arbitrage-Free Pricing Models

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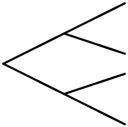
Outline

Option Pricing by Replication

- The original approach to option pricing, going back to Black, Scholes, and Merton, is to use a replication argument together with the Law of One Price.
- Consider a binomial model for the stock price

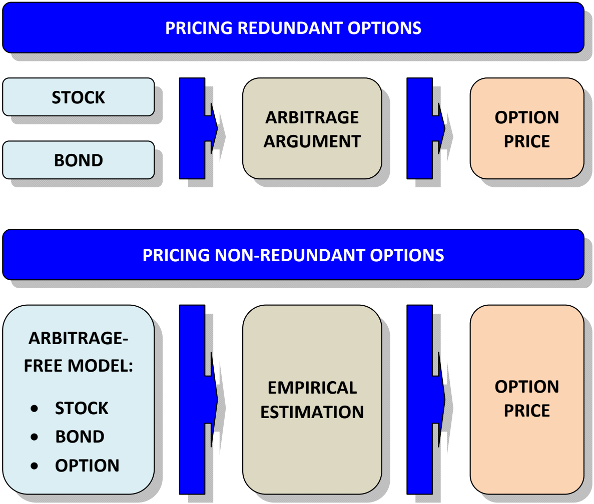


- Payoff of any option on the stock can be replicated by dynamic trading in the stock and the bond, thus there is a unique arbitrage-free option valuation.
- Problem solved?

Drawbacks of the Binomial Model

- The binomial model (and its variants) has a few issues.
- If the binomial depiction of market dynamics was accurate, all options would be redundant instruments. Is that realistic?
- Empirically, the model has problems: one should be able to replicate option payoffs perfectly in theory, that does not happen in reality.
- Why build models like the binomial model? Convenience. Unique option price by replication is a very appealing feature.
- How can one make the model more realistic, taking into account lack of perfect replication?

Arbitrage and Option Pricing

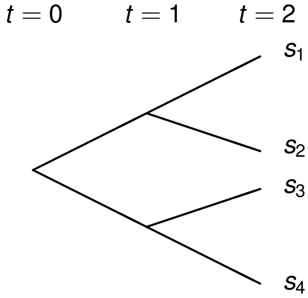


Arbitrage and Option Pricing

- Take an alternative approach to option pricing.
- Even when options cannot be replicated (options are not redundant), there should be no arbitrage in the market.
- The problem with non-redundant options is that there may be more than one value of the option price today consistent with no arbitrage.
- Change the objective: construct a tractable joint model of the primitive assets (stock, bond, etc.) and the options, which is
  - Free of arbitrage;
  - Conforms to empirical observations.
- When options are redundant, no need to look at option price data: there is a unique option price consistent with no arbitrage.
- When options are non-redundant, there may be many arbitrage-free option prices at each point in time, so we need to rely on historical option price data to help select among them.
- We know how to estimate dynamic models (MLE, QMLE, etc.). Need to learn how to build tractable arbitrage-free models.

Absence of Arbitrage

- Consider a finite-horizon discrete-time economy, time = $\{0, ..., T\}$.
- Assume a finite number of possible states of nature, $s = 1, ..., N$



Definition (Arbitrage)

Arbitrage is a feasible cash flow (generated by a trading strategy) which is non-negative in every state and positive with non-zero probability.

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- We often describe arbitrage as a strategy with no initial investment, no risk of a loss, and positive expected profit. It's a special case of the above definition.
- Absence of arbitrage implies the Law of One Price: two assets with the same payoff must have the same market price.
- Absence of arbitrage may be a very weak requirement in some settings and quite strong in others:
  - Equities: few securities, many states. Easy to avoid arbitrage.
  - Fixed income: many securities, few states. Not easy to avoid arbitrage.

Absence of Arbitrage

Fundamental Theorem of Asset Pricing (FTAP)

Proposition (FTAP)

Absence of arbitrage is equivalent to existence of a positive stochastic process $\{\pi_t(s) > 0\}$ *such that for any asset with price* P_t ($P_T = 0$) *and cash flow* D_t,

$$P_t(s) = E_t\left[\sum_{u=t+1}^{T} \frac{\pi_u(s)}{\pi_t(s)} D_u(s)\right]$$

or, in return form,

$$E_t\left[\frac{\pi_{t+1}(s)}{\pi_t(s)} R_{t+1}(s)\right] = 1, \quad R_{t+1}(s) = \frac{P_{t+1}(s) + D_{t+1}(s)}{P_t(s)}$$

- Stochastic process $\pi_t(s)$ is also called the state-price density (SPD).
- FTAP implies the Law of One Price.

Absence of Arbitrage

From SPD to No Arbitrage

- Prove one direction (the easier one): if SPD exists, there can be no arbitrage.
- Let W_t denote the portfolio value, $(\theta^1, ..., \theta^N)$ are holdings of risky assets $1, ..., N$.
- Manage the portfolio between $t = 0$ and $t = T$.
- An arbitrage is a strategy such that $W_0 \leqslant 0$ while $W_T \geqslant 0$, $W_T \neq 0$.
- The trading strategy is self-financing if it does not generate any cash in- or out-flows except for time 0 and T.
- Formal self-financing condition

$$W_{t+1} = \sum_i \theta^i_{t+1} P^i_{t+1} = \sum_i \theta^i_t (P^i_{t+1} + D^i_{t+1})$$

Absence of Arbitrage

From SPD to No Arbitrage

- Show that self-financing implies that $\pi_t W_t = \mathrm{E}_t[\pi_{t+1} W_{t+1}]$:

$$\mathrm{E}_t[\pi_{t+1} W_{t+1}] = \mathrm{E}_t[\pi_{t+1} \sum_i \theta^i_{t+1} P^i_{t+1}] \overset{\text{self financing}}{=} \mathrm{E}_t\left[\pi_{t+1} \sum_i \theta^i_t (P^i_{t+1} + D^i_{t+1})\right]$$
$$\overset{\text{FTAP}}{=} \sum_i \theta^i_t \pi_t P^i_t = \pi_t W_t$$

- Start at 0 and iterate forward

$$\begin{aligned}\pi_0 W_0 &= \mathrm{E}_0[\pi_1 W_1] = \mathrm{E}_0[\mathrm{E}_1[\pi_2 W_2]] \\ &= \mathrm{E}_0[\pi_2 W_2] = \dots \\ &= \mathrm{E}_0[\pi_T W_T]\end{aligned}$$

- Given that the SPD is positive, it is impossible to have $W_0 \leqslant 0$ while $W_T \geqslant 0$, $W_T \neq 0$. Thus, there can be no arbitrage.

Absence of Arbitrage

Fundamental Theorem of Asset Pricing (FTAP)

- When there exists a full set of state-contingent claims (markets are complete), there is a unique SPD consistent with absence of arbitrage:

$\pi_t(s)$ is the price of a state-contingent claim paying \$1 in state s at time t, normalized by the probability of that state, p_t.

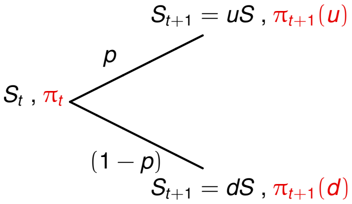
- The reverse is also true: if there exists only one SPD, all options are redundant.
- When there are fewer assets than states of nature, there can be many SPDs consistent with no arbitrage.
- FTAP says that if there is no arbitrage, there must be at least one way to introduce a consistent system of positive state prices.
- We drop explicit state-dependence and write π_t instead of $\pi_t(s)$.

Example: Binomial Tree

SPD

- Options are redundant: any payoff can be replicated by dynamic trading.
- FTAP implies that

$$
\begin{aligned}
p\frac{\pi_{t+1}(u)}{\pi_t}(1+r) + (1-p)\frac{\pi_{t+1}(d)}{\pi_t}(1+r) &= 1 \\
p\frac{\pi_{t+1}(u)}{\pi_t}u + (1-p)\frac{\pi_{t+1}(d)}{\pi_t}d &= 1
\end{aligned}
$$



- SPD is unique up to normalization $\pi_0 = 1$:

$$
\begin{aligned}
\frac{\pi_{t+1}(u)}{\pi_t} &= \frac{1}{p(1+r)}\frac{1+r-d}{u-d} \\
\frac{\pi_{t+1}(d)}{\pi_t} &= \frac{1}{(1-p)(1+r)}\frac{u-(1+r)}{u-d}
\end{aligned}
$$

Arbitrage-Free Models using SPD

Discounted Cash Flow Model (DCF)

Algorithm: A DCF Model

1. Specify the process for cash flows, D_t.
2. Specify the SPD, π_t.
3. Derive the asset price process as

$$P_t = \mathrm{E}_t\left[\sum_{u=t+1}^{T} \frac{\pi_u}{\pi_t} D_u\right] \tag{DCF}$$

- To make this practical, need to learn how to parameterize SPDs in step (2), so that step (3) can be performed efficiently.
- Can use discrete-time conditionally Gaussian processes.
- SPDs are closely related to risk-neutral pricing measures. Useful for building intuition and for computations.

SPD and the Risk Premium

- Let R_{t+1} be the gross return on a risky asset between t and $t+1$:

$$\mathrm{E}_t\left[\frac{\pi_{t+1}}{\pi_t} R_{t+1}\right] = 1$$

$$\mathrm{E}_t\left[\frac{\pi_{t+1}}{\pi_t} (1+r_t)\right] = 1$$

- Using the definition of covariance,

Conditional Risk Premium and SPD Beta

$$\text{Risk Premium}_t \equiv \mathrm{E}_t[R_{t+1}] - (1+r_t) = -(1+r_t)\mathrm{Cov}_t\left(R_{t+1}, \frac{\pi_{t+1}}{\pi_t}\right)$$

SPD and CAPM

- CAPM says that risk premia on all stocks must be proportional to their market betas.
- CAPM can be re-interpreted as a statement about the SPD pricing all assets.
- Assume that

$$\pi_{t+1}/\pi_t = a - bR^M_{t+1}$$

  where R^M is the return on the market portfolio. (The above formula may be viewed as approximation, if it implies negative values of π).
- Using the general formula for the risk premium, for any stock j,

$$\mathrm{E}_t\left[R^j_{t+1} - (1 + r_t)\right] = const \times \mathrm{Cov}_t(R^j_{t+1}, R^M_{t+1})$$

- The above formula works for any asset, including the market return. Use this to find the constant:

$$\mathrm{E}_t\left[R^j_{t+1} - (1 + r_t)\right] = \mathrm{E}_t\left[R^M_{t+1} - (1 + r_t)\right] \frac{\mathrm{Cov}_t(R^j_{t+1}, R^M_{t+1})}{\mathrm{Var}_t(R^M_{t+1})}$$

SPD and Multi-Factor Models

- Alternative theories (e.g., APT), imply that there are multiple priced factors in returns, not just the market factor.
- Multi-factor models are commonly used to describe the cross-section of stock returns (e.g., the Fama-French 3-factor model).
- Assume that

$$\pi_{t+1}/\pi_t = a + b_1 F_{t+1}^1 + ...b_K F_{t+1}^K$$

  where F^k, $k = 1, ..., K$ are K factors. Factors may be portfolio returns, or non-return variables (e.g., macro shocks).
- Then risk premia on all stocks have factor structure

$$\mathrm{E}_t \left[R_{t+1}^j - (1 + r_t) \right] = \sum_{k=1}^{K} \lambda_k \mathrm{Cov}_t \left(R_{t+1}^j, F_{t+1}^k \right)$$

- Factor models are simply statements about the factor structure of the SPD.

SPDs and Risk-Neutral Pricing

- One can build models by specifying the SPD and computing all asset prices.
- It is typically more convenient to use a related construction, called risk-neutral pricing.
- Risk-neutral pricing is a mathematical construction. It is often convenient and adds something to our intuition.

Risk-Neutral Measures

- The DCF formulation with an SPD is mathematically equivalent to a change of measure from the physical probability measure to the risk-neutral measure. The risk-neutral formulation offers a useful and tractable alternative to the DCF model.
- Let **P** denote the physical probability measure (the one behind empirical observations), and **Q** denote the risk-neutral measure. **Q** is a mathematical construction used for pricing and only indirectly connected to empirical data.
- Let B_t denote the value of the risk-free account:

$$B_t = \prod_{u=0}^{t-1} (1 + r_u)$$

where r_u is the risk-free rate during the period $[u, u+1)$.

Risk-Neutral Measures

- **Q** is a probability measure under which

$$P_t = \mathrm{E}_t^{\mathbf{P}}\left[\sum_{u=t+1}^{T} \frac{\pi_u}{\pi_t} D_u\right] = \mathrm{E}_t^{\mathbf{Q}}\left[\sum_{u=t+1}^{T} \frac{B_t}{B_u} D_u\right], \quad \text{for any asset with cash flow } D$$

- Under **Q**, the standard DCF formula holds.
- Under **Q**, expected returns on all assets are equal to the risk-free rate:

$$\mathrm{E}_t^{\mathbf{Q}}\left[R_{t+1}\right] = 1 + r_t$$

- If **Q** has positive density with respect to **P**, there is no arbitrage.
- There may exist multiple risk-neutral measures.
- **Q** is also called an equivalent martingale measure (EMM).

SPD and Change of Measure

- Construct risk-neutral probabilities from the SPD.
- Consider our tree-model of the market and let $\mathcal{C}(\nu_t)$ denote the set of time-$(t+1)$ nodes which are children of node ν_t.
- Define numbers $q(\nu_{t+1})$ for all nodes $\nu_{t+1} \in \mathcal{C}(\nu_t)$ by the formula

$$q(\nu_{t+1}) = (1 + r_t) q(\nu_t) \frac{\pi(\nu_{t+1}) p(\nu_{t+1})}{\pi(\nu_t) p(\nu_t)}$$

- Recall that the ratio $p(\nu_{t+1})/p(\nu_t)$ is the node-ν_t conditional probability of ν_{t+1}.
- $q(\nu_{t+1}) > 0$ and

$$\sum_{\nu_{t+1} \in \mathcal{C}(\nu_t)} q(\nu_{t+1}) = q(\nu_t) \sum_{\nu_{t+1} \in \mathcal{C}(\nu_t)} (1 + r_t) \frac{\pi(\nu_{t+1}) p(\nu_{t+1})}{\pi(\nu_t) p(\nu_t)}$$

$$= q(\nu_t) \mathrm{E}_t^{\mathbf{P}} \left[\frac{\pi(\nu_{t+1})}{\pi(\nu_t)} (1 + r_t) \right] = q(\nu_t)$$

SPD and Change of Measure

- $q(\nu_t)$ define probabilities. Are these risk-neutral probabilities?
- For any asset i,

$$
\begin{aligned}
\mathrm{E}_t^{\mathbf{Q}}\left[\frac{1}{1+r_t}R_{t+1}^i\right] &= \sum_{\nu_{t+1}\in\mathcal{C}(\nu_t)} \frac{q(\nu_{t+1})}{q(\nu_t)}\frac{1}{1+r_t}R_{t+1}^i \\
&= \sum_{\nu_{t+1}\in\mathcal{C}(\nu_t)} (1+r_t)\frac{\pi(\nu_{t+1})p(\nu_{t+1})}{\pi(\nu_t)p(\nu_t)}\frac{1}{1+r_t}R_{t+1}^i \\
&= \sum_{\nu_{t+1}\in\mathcal{C}(\nu_t)} \frac{p(\nu_{t+1})}{p(\nu_t)}\frac{\pi(\nu_{t+1})}{\pi(\nu_t)}R_{t+1}^i \\
&= \mathrm{E}_t^{\mathbf{P}}\left[\frac{\pi_{t+1}}{\pi_t}R_{t+1}^i\right] = 1
\end{aligned}
$$

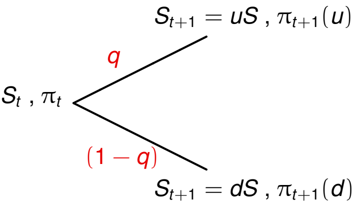
- Conclude that $q(\nu_t)$ define risk-neutral probabilities.

Example: Binomial Tree

Risk-Neutral Measure

- FTAP implies that

$$qu+(1-q)d = 1+r_t \quad \Rightarrow \quad q = \frac{1+r-d}{u-d}$$



- Alternatively, compute transition probability under **Q** using the SPD π:

$$q = p(1+r_t)\frac{\pi_{t+1}(u)}{\pi_t}$$
$$= p(1+r_t)\frac{1}{p(1+r_t)}\frac{1+r_t-d}{u-d} = \frac{1+r_t-d}{u-d}$$

Normality-Preserving Change of Measure

Results

- Under $\mathbf{P}$, $\varepsilon^{\mathbf{P}} \sim \mathcal{N}(0,1)$. Define a new measure $\mathbf{Q}$, such that under $\mathbf{Q}$, $\varepsilon^{\mathbf{P}} \sim \mathcal{N}(-\eta, 1)$.
- Let $\xi = \frac{d\mathbf{Q}}{d\mathbf{P}}$. Then,

$$\xi(\varepsilon^{\mathbf{P}}) = \exp\left(-\frac{(\varepsilon^{\mathbf{P}}+\eta)^2}{2} + \frac{(\varepsilon^{\mathbf{P}})^2}{2}\right) = \exp\left(-\eta\varepsilon^{\mathbf{P}} - \frac{\eta^2}{2}\right)$$

The change of measure is given by a log-normal random variable $\xi(\varepsilon^{\mathbf{P}})$ serving as the density of the new measure.

Normality-Preserving Change of Measure

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = e^{-\eta\varepsilon^{\mathbf{P}} - \eta^2/2} \Rightarrow \varepsilon^{\mathbf{Q}} = \varepsilon^{\mathbf{P}} + \eta \sim \mathcal{N}(0,1) \text{ under } \mathbf{Q}$$

Price of Risk

- When **P** and **Q** are both Gaussian, we can define the *price of risk* (key notion in continuous-time models).
- Consider an asset with gross return

$$R_{t+1} = \exp\left(\mu_t - \sigma_t^2/2 + \sigma_t \varepsilon_{t+1}^{\mathbf{P}}\right), \quad \mathrm{E}_t[R_{t+1}] = \exp(\mu_t)$$

  where $\varepsilon_{t+1}^{\mathbf{P}} \sim \mathcal{N}(0,1)$, IID, under the physical **P**-measure.
- Let the short-term risk-free interest rate be

$$\exp(r_t) - 1$$

- Let the SPD be

$$\pi_{t+1} = \pi_t \exp\left(-r_t - \eta_t^2/2 - \eta_t \varepsilon_{t+1}^{\mathbf{P}}\right)$$

- Recall

$$\varepsilon_t^{\mathbf{Q}} = \varepsilon_t^{\mathbf{P}} + \eta_t$$

- Under **Q**, the return distribution becomes

$$R_{t+1} = \exp\left(\mu_t - \sigma_t^2/2 - \sigma_t \eta_t + \sigma_t \varepsilon_{t+1}^{\mathbf{Q}}\right)$$

  where $\varepsilon_{t+1}^{\mathbf{Q}} \sim \mathcal{N}(0,1)$, IID, under the **Q**-measure.

Price of Risk

- Under **Q**,

$$R_{t+1} = \exp\left(\mu_t - \sigma_t^2/2 - \sigma_t\eta_t + \sigma_t\varepsilon_{t+1}^{\mathbf{Q}}\right), \quad \varepsilon_{t+1}^{\mathbf{Q}} \sim \mathcal{N}(0,1)$$

- By definition of the risk-neutral probability measure,

$$\mathrm{E}_t^{\mathbf{Q}}[R_{t+1}] = \exp(r_t) \quad \Rightarrow \quad \mu_t - \sigma_t\eta_t = r_t$$

- The risk premium (measured using log expected gross returns) equals

$$\mu_t - r_t = \sigma_t\eta_t$$

  σ_t is the quantity of risk,
  η_t is the price of risk.

- Models with time-varying price of risk, η_t, exhibit return predictability.

Overview

- As an example of how risk-neutral pricing is used, we consider the problem of option pricing when the underlying asset exhibits stochastic volatility.
- The benchmark model is the Black-Scholes model.
- Stock (underlying) volatility in the B-S model is constant.
- Empirically, the B-S model is rejected: the implied volatility is not the same for options with different strikes.
- There are many popular generalizations of the B-S model. We explore the model with an EGARCH volatility process.
- The EGARCH model addresses some of the empirical limitations of the B-S model.

The Black-Scholes Model

- Consider a stock with price S_t, no dividends. Assume that

$$\frac{S_{t+1}}{S_t} = \exp\left(\mu - \sigma^2/2 + \sigma\varepsilon^{\mathbf{P}}_{t+1}\right),$$

where $\varepsilon^{\mathbf{P}}_{t+1} \sim \mathcal{N}(0,1)$, IID, under the physical **P**-measure.

- Assume that the short-term interest rate is constant.
- Assume that under the **Q**-measure,

$$\frac{S_{t+1}}{S_t} = \exp\left(r - \sigma^2/2 + \sigma\varepsilon^{\mathbf{Q}}_{t+1}\right), \quad \varepsilon^{\mathbf{Q}}_{t+1} \sim \mathcal{N}(0,1), \text{ IID}$$

- The time-t price of *any* European option on the stock, with a payoff $C_T = H(S_T)$, is given by

$$C_t = \mathrm{E}^{\mathbf{Q}}_t\left[e^{-r(T-t)} H(S_T)\right]$$

- This is an arbitrage-free model. Prices of European call and put options are given by the Black-Scholes formula.

Implied Volatility

- The Black-Scholes model expresses the price of the option as a function of the parameters and the current stock price.
- European Call option price

$$C(S_t, K, r, \sigma, T)$$

- Implied volatility $\widehat{\sigma}_i$ of a Call option with strike K_i and time to maturity T_i is defined by

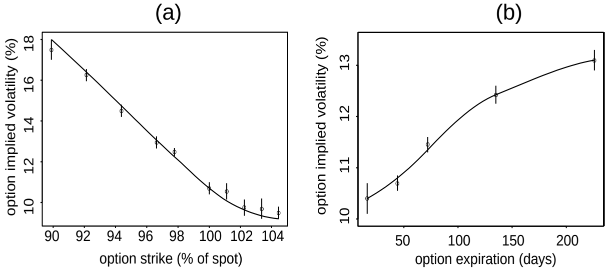
$$C_i = C(S_t, K_i, r, \widehat{\sigma}_i, T_i)$$

- Implied volatility reconciles the observed option price with the B-S formula.
- Option prices are typically quoted in terms of implied volatilities.

Implied Volatility Smile (Smirk)

- If option prices satisfied the B-S assumptions, all implied volatilities would be the same, equal to σ, the volatility of the underlying price process.
- Empirically, implied volatilities depend on the strike and time to maturity.

FIGURE 2. Implied Volatilities of S&P 500 Options on May 5, 1993



Source: E. Derman, I. Kani, 1994, The Volatility Smile and Its Implied Tree, *Quantitative Strategies Research Notes*, Goldman Sachs

EGARCH Model

- Consider a model of stock returns with EGARCH volatility process.
- Assume the stock pays no dividends, and short-term interest rate is constant.
- Stock returns are *conditionally* log-normally distributed under the **Q**-measure

$$\ln \frac{S_t}{S_{t-1}} = r - \frac{\sigma_{t-1}^2}{2} + \sigma_{t-1}\varepsilon_t^{\mathbf{Q}}, \quad \varepsilon_t^{\mathbf{Q}} \sim \mathcal{N}(0,1), \text{ IID}$$

- Conditional expected gross return on the underlying asset equals $\exp(r)$.
- Conditional volatility σ_t follows an EGARCH process under **Q**

$$\ln\left(\sigma_t^2\right) = a_0 + b_1 \ln\left(\sigma_{t-1}^2\right) + \theta\varepsilon_{t-1}^{\mathbf{Q}} + \gamma\left(\left|\varepsilon_{t-1}^{\mathbf{Q}}\right| - \sqrt{\frac{2}{\pi}}\right)$$

- Option prices can be computed using the risk-neutral valuation formula.

Option Valuation by Monte Carlo

- We use Monte Carlo simulation to compute option prices.
- Using the valuation formula

$$C_t = \mathrm{E}_t^{\mathbf{Q}}\left[e^{-r(T-t)} H(S_T)\right]$$

Call option price can be estimated by simulating N trajectories of the underlying asset S_u^n, $n = 1, ..., N$, under **Q**, and averaging the discounted payoff

$$C_t \approx \frac{1}{N} \sum_{n=1}^{N} e^{-r(T-t)} \max(S_T^n - K, 0)$$

- Resulting option prices are arbitrage-free because they satisfy the risk-neutral pricing relationship.

Simulation

- Use one-week time steps.
- Calibrate the parameters using the estimates in Day and Lewis (1992, Table 3)

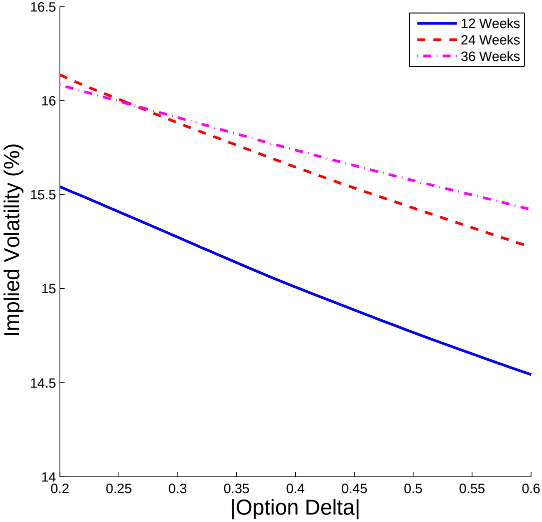
a_0	b_1	θ	γ
-3.620	0.529	-0.273	0.357

- Calibrate the interest rate

$$\exp(52 \times r) = \exp(0.05)$$

- Start all N trajectories with the same initial stock price and the same initial volatility, $\sigma_0 = 15.5\%/\sqrt{52}$.
- Compute implied volatilities for Call/Put options with different strikes.
- Plot implied volatilities against Black-Scholes deltas of Put options ($\Delta_t = \partial P_t / \partial S_t$).

Volatility Smile



Futures

- We want to build an arbitrage-free model of futures prices.
- In case of deterministic interest rates and costless storage of the underlying asset, futures price is the same as the forward price.
- In most practical situations, storage is not costless, so simple replication arguments are not sufficient to derive futures prices.
- Want to model futures prices for multiple maturities in an arbitrage-free framework.

Risk-Neutral Pricing

- Φ_t^T is the time-t futures price for a contract maturing at T.
- Futures are continuously settled, so the holder of the long position collects

$$\Phi_t^T - \Phi_{t-1}^T$$

  each period. Continuous settlement reduces the likelihood of default.
- The market value of the contract is always equal to zero.
- In the risk-neutral pricing framework

$$\mathrm{E}_t^{\mathbf{Q}}\left[\sum_{u=t+1}^{T} \frac{B_t}{B_u}\left(\Phi_u^T - \Phi_{u-1}^T\right)\right] = 0 \quad \text{for all } t$$

- We conclude (using backwards induction and iterated expectations) that the futures prices must satisfy

$$\mathrm{E}_t^{\mathbf{Q}}\left[\Phi_{t+1}^T - \Phi_t^T\right] = 0 \quad \text{for all } t$$

  and thus

$$\Phi_t^T = \mathrm{E}_t^{\mathbf{Q}}\left[\Phi_T^T\right] = \mathrm{E}_t^{\mathbf{Q}}\left[S_T\right]$$

  where S_T is the spot price at T.

AR(1) Spot Price

- Suppose that the spot price follows an AR(1) process under the **P**-measure

$$S_t - \overline{S} = \theta(S_{t-1} - \overline{S}) + \sigma\varepsilon_t^{\mathbf{P}}, \quad \varepsilon_t^{\mathbf{P}} \sim \mathcal{N}(0,1), \text{ IID}$$

- Assume that the risk-neutral **Q**-measure is related to the physical **P**-measure by the state-price density

$$\frac{\pi_{t+1}}{\pi_t} = \exp\left(-r_t - \frac{\eta^2}{2} - \eta\varepsilon_{t+1}^{\mathbf{P}}\right)$$

- Market price of risk η is constant.
- To compute futures prices, use $\varepsilon_t^{\mathbf{Q}} = \varepsilon_t^{\mathbf{P}} + \eta$.

AR(1) Spot Price

- Under the risk-neutral measure, spot price follows

$$S_t = \left(\overline{S} - \frac{\eta\sigma}{1-\theta}\right) + \theta\left[S_{t-1} - \left(\overline{S} - \frac{\eta\sigma}{1-\theta}\right)\right] + \sigma\varepsilon_t^{\mathbf{Q}}$$

- Define a new constant

$$\overline{S}^{\mathbf{Q}} \equiv \overline{S} - \frac{\eta\sigma}{1-\theta}$$

  Then

$$S_t - \overline{S}^{\mathbf{Q}} = \theta\left(S_{t-1} - \overline{S}^{\mathbf{Q}}\right) + \sigma\varepsilon_t^{\mathbf{Q}}$$

- Under **Q**, spot price is still AR(1), same mean-reversion rate, but different long-run mean.

AR(1) Spot Price

- To compute the futures price, iterate the AR(1) process forward

$$
\begin{aligned}
S_{t+1} - \overline{S}^{\mathbf{Q}} &= \theta \left(S_t - \overline{S}^{\mathbf{Q}} \right) + \sigma \varepsilon^{\mathbf{Q}}_{t+1} \\
S_{t+2} - \overline{S}^{\mathbf{Q}} &= \theta \left(S_{t+1} - \overline{S}^{\mathbf{Q}} \right) + \sigma \varepsilon^{\mathbf{Q}}_{t+2} \\
&= \theta^2 \left(S_t - \overline{S}^{\mathbf{Q}} \right) + \sigma \varepsilon^{\mathbf{Q}}_{t+2} + \theta \sigma \varepsilon^{\mathbf{Q}}_{t+1} \\
&\vdots \\
S_{t+n} - \overline{S}^{\mathbf{Q}} &= \theta^n \left(S_t - \overline{S}^{\mathbf{Q}} \right) + \sigma \varepsilon^{\mathbf{Q}}_{t+n} + \ldots + \theta^{n-2} \sigma \varepsilon^{\mathbf{Q}}_{t+2} + \theta^{n-1} \sigma \varepsilon^{\mathbf{Q}}_{t+1}
\end{aligned}
$$

- We conclude that

$$
\Phi_t^T = \mathrm{E}_t^{\mathbf{Q}} \left[S_T \right] = \overline{S}^{\mathbf{Q}} \left(1 - \theta^{T-t} \right) + \theta^{T-t} S_t
$$

- Futures prices of various maturities given by the above model do not admit arbitrage.

Expected Gain on Futures Positions

- What is the expected gain on a long position in a futures contract?
- Under **Q**, the expected gain is zero:

$$\mathrm{E}_t^{\mathbf{Q}}\left[\left(\Phi_{t+1}^T - \Phi_t^T\right)\right] = 0 \quad \text{for all } t$$

- Futures contracts provides exposure to risk, $\varepsilon^{\mathbf{P}}$. This risk is compensated, with the market price of risk η.

$$\Phi_{t+1}^T - \Phi_t^T = \sigma\theta^{T-t-1}\varepsilon_{t+1}^{\mathbf{Q}} = \eta\sigma\theta^{T-t-1} + \sigma\theta^{T-t-1}\varepsilon_{t+1}^{\mathbf{P}}$$

(Use $\varepsilon_{t+1}^{\mathbf{Q}} = \eta + \varepsilon_{t+1}^{\mathbf{P}}$)

- Under **P**, expected gain is non-zero, because $\varepsilon^{\mathbf{Q}}$ has non-zero mean under **P**

$$\mathrm{E}_t^{\mathbf{P}}\left[\left(\Phi_{t+1}^T - \Phi_t^T\right)\right] = \eta\sigma\theta^{T-t-1} \quad \text{for all } t$$

- Can estimate model parameters, including η, from historical futures prices.

Summary

- Existence of SPD or risk-neutral probability measure guarantees absence of arbitrage.
- Factor pricing models, e.g., CAPM, are models of the SPD.
- Can build consistent models of multiple options by specifying the risk-neutral dynamics of the underlying asset.
- Black-Scholes model, volatility smiles, and stochastic volatility models.

Readings

- Back 2005, Chapter 1.
- E. Derman, I. Kani, 1994, The Volatility Smile and Its Implied Tree, *Quantitative Strategies Research Notes*, Goldman Sachs.
- T. Day, C. Lewis, 1992, Stock Market Volatility and the Information Content of Stock Index Options, *Journal of Econometrics* 52, 267-287.

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