17.812J / 14.296J Collective Choice I
Fall 2008

Problem 3.B

This problems builds on Austen-Smith(1987) and Cox(1987). Consider the following multi-party election game:

There are three voters, where the set of voters is denoted as $N = \{1, 2, 3\}$. Let $\mathbf{X}$ be the set of feasible policies. Assume that it is a compact interval $[-K, K]$ where $K > 2$. Let any voter i have quadratic utility

$$u_i(x) = -(x_i - x)^2 \tag{3}$$

where x is the policy implemented and x_i is the ideal point of voter i. For simplicity, assume that $x_1 = -1$, $x_2 = 0$, $x_3 = 1$.

There are three candidates, a, b and c, competing for election. Let $\mathbf{C}$ be the set of discrete subsets of $\mathbf{X}$. Any $C \in \mathbf{C}$ with $|C| \leq 3$ is a slate of candidate platforms. Each slate specifies a platform y_k for candidate k. We consider two objective functions for the candidates. In one case, candidates maximize their vote share. In the other, they maximize the probability of winning, where the winner of the election is determined by plurality rule. Formally, write $p_j(C; \sigma)$ as the probability that j wins the election given candidate platforms (C) and voting strategies ($\sigma = (\sigma_i)_{i \in N}$). We have that

$$p_j(C; \sigma) = \begin{cases} 1 & \text{if } n_j > n_k \ \forall k \neq j \\ 1/2 & \text{if } n_j = n_k > n_l \ \text{ for some } k, l \in C \\ 1/3 & \text{if } n_j = n_k = n_l \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where n_k is the number of votes for candidate k. Moreover, let us call $W(C; \sigma)$ the set of possible winning platforms, ie

$$W(C; \sigma) = \{y_k \in C \mid \ p_k(C; \sigma) > 0\} \tag{5}$$

As a result of the election, voter i's expected utility is:

$$U_i(C; \sigma) \equiv \sum_j p_j(C; \sigma_i, \sigma_{-i}) u_i(y_j) \tag{6}$$

The timing of the game is as follows:

1. Candidates simultaneously announce a policy platform
2. Citizens vote

Throughout, we are looking to characterize equilibrium outcomes when strategies form a (Subgame-perfect) Nash Equilibrium. ie candidate k's strategy y_k is optimal given strategies $\mathbf{y}_{-k}$ by the other two candidates and voting strategies $\boldsymbol{\sigma}$ by the voters. i's voting strategy, $\boldsymbol{\sigma}_i$, is optimal given voting strategies $\boldsymbol{\sigma}_{-i}$ by the other voters and the slate of platforms C.

Question 1

Assume first that candidates maximize the probability of a win.

1.1) Prove the following statement: $\forall x \in \mathbf{X}, \exists C^{NE}, \boldsymbol{\sigma}^{NE} | \ x \in W(C^{NE}, \boldsymbol{\sigma}^{NE})$, where $(C^{NE}, \boldsymbol{\sigma}^{NE})$ is a slate and a vector of pure voting strategies that form a Nash Equilibrium.

Now assume that candidates maximize their vote share.

1.2) Prove the same statement.

Question 2

Restrict strategies to be sincere with respect to C, ie

$$\sigma_i(C) = y_k \Rightarrow \neg\{\exists y'_k \in C \setminus \{y_k\} | u_i(y'_k) > u_i(y_k)\} \tag{7}$$

and let each voter randomize with equal probability if there is indifference among his/her favorite options.

Assume first that candidates maximize the probability of a win.

2.1) Does Lemma 1 of Cox(1987) hold? Why or why not?

2.2) Does the set of equilibrium outcomes shrink (relative to Question 1)? Find the bounds on such a set.

Now assume that candidates maximize their vote share.

2.3) Does Lemma 1 of Cox(1987) hold? Why or why not?

2.4) Does the set of equilibrium outcomes shrink (relative to Question 1)? Find the bounds on such a set.

Question 3

Say that σ_i^T is *weakly dominated* for i given C if $\exists \sigma_i'$ such that

$$U_i(C; \sigma_i^T, \sigma_{-i}) \leq U_i(C; \sigma_i', \sigma_{-i}) \;\; \forall \sigma_{-i} \tag{8}$$

with the inequality being strict for some σ_{-i}. Now restrict attention to strategies that are not weakly dominated.

Assume first that candidates maximize the probability of a win.

3.1) Does the set of equilibrium outcomes shrink relative to the answer of Question 1? relative to the answer of Question 2?

Now assume that candidates maximize their vote share.

3.2) Does the set of equilibrium outcomes shrink relative to the answer of Question 1? relative to the answer of Question 2?

Question 4

Now modify the game and allow individuals to collude so as to restrict candidate platforms. In particular, let the revised timing be as follows:

1. Candidates simultaneously announce a policy platform
2. Voters 'decide' whether to block the slate of platforms
3. If the slate is blocked, go back to 1. If the slate is accepted, citizens vote sincerely.

Consider any equilibrium $(A, \boldsymbol{\sigma}^S)$ where $\boldsymbol{\sigma}^S$ is a vector of sincere voting strategies as defined in Question 2. We will say that a slate A is blocked under plurality rule iff $\exists L, N \supseteq L$, and $\exists \sigma_L = (\sigma_i)_{i \in L}$ such that

$$U_i(A; \sigma_L, \sigma_{-L}^S) \geq U_i(A; \sigma_L^S, \sigma_{-L}^S) \; \forall i \in L \tag{9}$$

and $\exists i \in L$ s.t. (9) holds strictly. If a slate is blocked forever, let the utility of each candidate be some value $B < 0$.

Assume first that candidates maximize the probability of a win.

4.1) Does the set of equilibrium outcomes shrink relative to the answer Question 2? Please comment.

Now assume that candidates maximize their vote share:

4.2) Does the set of equilibrium outcomes shrink relative to the answer of Question 2? Please comment.

References

[1] Austen-Smith, David. 1987. "Electing Legislatures" Caltech Social Science Working paper 644 (stellar)

[2] Besley, Timothy and Stephen Coate (1997). 'An Economic Model of Representative Democracy', *The Quarterly Journal of Economics*, 112(1): 85-114

[3] Cox, Gary W. 1987. "Electoral Equilibrium under Alternative Voting Institutions." *American Journal of Political Science* 31(1): 82-108 (JSTOR)