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6.055J / 2.038J The Art of Approximation in Science and Engineering
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$$r = \sqrt{\frac{3\sqrt{3}}{4\pi}}.$$

The bisection path is one-sixth of a circle, so its length is

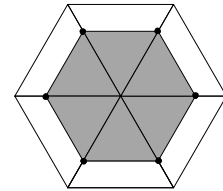
$$l = \frac{2\pi r}{6} = \frac{\pi}{3} \sqrt{\frac{3\sqrt{3}}{4\pi}} = \sqrt{\frac{\pi\sqrt{3}}{12}}.$$

The best previous candidate (the first picture) has length $1/\sqrt{2} = 0.707\dots$. Does the mess of π and square roots produce a shorter path? Roll the drums...:

$$l = 0.67338\dots,$$

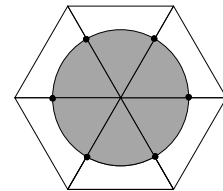
which is less than $1/\sqrt{2}$. So the circular arc is the best bisection path *so far*. However, is it the best among all possible paths? The arc-length calculation for the circle is messy, and most other paths do not even have a closed form for their arc lengths.

Instead of making elaborate calculations on every path, of which there are many, try **symmetry**, which is the mathematical principle for the three methods in this part of the book. To use symmetry, replicate the triangle six times to make a hexagon, thereby replicating the candidate path as well.



Here is the result of replicating the first candidate where the bisection line goes straight across. The original triangle becomes the large hexagon, and the enclosed half-triangle becomes a smaller hexagon having one-half the area of the large hexagon.

Compare that picture with the result of replicating the circular-arc bisection. The large hexagon is the same as for the last replication, but now the bisected area replicates into a circle. Which path has the shorter perimeter, the shaded hexagon or this circle? The **isoperimetric theorem** says that among all shapes with the same area the circle has the smallest perimeter. Since the circle and the smaller hexagon enclose the same area – which is three times the area of one triangle – the circle has a smaller perimeter than the hexagon, and it has a smaller perimeter than the result of replicating any other bisecting path. So the circular arc is the solution.

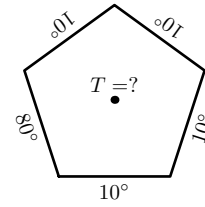


The lesson of this example is that symmetry can remove complexity. The complexity in this problem comes from the edges of the triangle: How much of each edge should be part of the bisected shape? Different paths use different amounts of each edge, and there's no obvious way to deduce the correct amounts. After making the figure symmetric by replicating the triangle into a hexagon, the edges become irrelevant. In the symmetric figure, the question simplifies to finding the shortest path that encloses one-half of the hexagon.

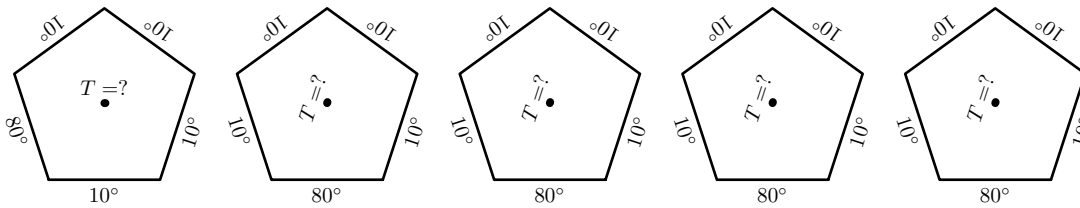
4.3 Heat flow

Here is a metal sheet in the shape of a regular pentagon with the sides held at fixed temperatures. What is the temperature at the center of the pentagon?

This problem is difficult to solve analytically because heat flow is described by a second-order partial differential equation, and this equation has simple solutions only for a few simple boundaries. A pentagon is, alas, not one of those boundaries. Symmetry, however, makes the solution flow.



The symmetry operation is rotation because the pentagon's orientation is an arbitrary choice. Nature, in the person of the heat equation, does not care how we point our coordinate systems. So these five orientations of the sheet behave the same:



Now stack these sheets (mentally), adding their temperatures that lie on top of each other to get the temperature profile of a new metal sheet. For the new sheet, each edge has temperature

$$T_{\text{edge}} = 80^\circ + 10^\circ + 10^\circ + 10^\circ + 10^\circ = 120^\circ.$$

Therefore, the entire sheet is at 120° .

Since the symmetry operation is a rotation (by 72°) about the center, the centers overlap when the plates are (mentally) stacked one on top of the other. Furthermore, heat flow is proportional to temperature difference – i.e. heat flow is a linear process – so the temperature in the interior of the combined plate is the sum of the five corresponding interior temperatures. Since the stacked plate has a temperature of 120° throughout it, and the centers of the five subsheets align on top of each other, each center is at $T = 120^\circ/5 = 24^\circ$.

4.4 Looking forward

The next three chapters use this aspect of symmetry – finding and removing fake complexity – to develop three techniques: proportional reasoning, conservation, and dimensional analysis.