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6.055J / 2.038J The Art of Approximation in Science and Engineering
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The other possibility is to add quantum mechanics, which was developed to solve fundamental problems like the existence of matter. The physics of quantum mechanics is complicated, but its effect on dimensional analyses is simple: It contributes a new constant of nature \hbar whose dimensions are those of angular momentum. Angular momentum is mvr , so

$$[\hbar] = \text{ML}^2\text{T}^{-1}.$$

The \hbar might save the day. There are now two quantities containing time dimensions. Since $e^2/4\pi\epsilon_0$ has T^{-2} and \hbar has T^{-1} , the ratio $\hbar^2/(e^2/4\pi\epsilon_0)$ contains no time dimensions. Since

$$\left[\frac{\hbar^2}{e^2/4\pi\epsilon_0} \right] = \text{ML},$$

a dimensionless group is

$$\frac{\hbar^2}{a_0 m_e (e^2/4\pi\epsilon_0)}$$

It turns out that all dimensionless groups can be formed from this group. So, as in the spring-mass example, the only possible true statement involving this group is

$$\frac{\hbar^2}{a_0 m_e (e^2/4\pi\epsilon_0)} = \text{dimensionless constant.}$$

Therefore, the size of hydrogen is

$$a_0 \sim \frac{\hbar^2}{m_e (e^2/4\pi\epsilon_0)}.$$

Putting in values for the constants gives

$$a_0 \sim 0.5\text{\AA} = 0.5 \cdot 10^{-10} \text{ m.}$$

It turns out that the missing dimensionless constant is 1, so the dimensional analysis has given the exact answer.

7.4.2 Atomic sizes and substance densities

Hydrogen has a diameter of 1\AA . A useful consequence is the rule of thumb is that a typical interatomic spacing is 3\AA . This approximation gives a reasonable approximation for the densities of substances, as this section explains.

<i>Var</i>	<i>Dim</i>	What
a_0	L	size
$e^2/4\pi\epsilon_0$	ML^3T^{-2}	
m_e	M	electron mass
\hbar	ML^2T^{-1}	quantum

Let A be the atomic mass of the atom; it is (roughly) the number of protons and neutrons in the nucleus. Although A is called a mass, it is dimensionless. Each atom occupies a cube of side length $a \sim 3 \text{ \AA}$, and has mass Am_{proton} . The density of the substance is

$$\rho = \frac{\text{mass}}{\text{volume}} \sim \frac{Am_{\text{proton}}}{(3 \text{ \AA})^3}.$$

You do not need to remember or look up m_{proton} if you multiply this fraction by unity in the form of N_A/N_A , where N_A is Avogadro's number:

$$\rho \sim \frac{Am_{\text{proton}}N_A}{(3 \text{ \AA})^3 \times N_A}.$$

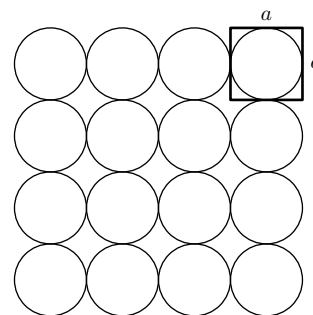
The numerator is $A \text{ g}$, because that is how N_A is defined. The denominator is

$$3 \cdot 10^{-23} \text{ cm}^3 \times 6 \cdot 10^{23} = 18.$$

So instead of remembering m_{proton} , you need to remember N_A . However, N_A is more familiar than m_{proton} because N_A arises in chemistry and physics. Using N_A also emphasizes the connection between microscopic and macroscopic values. Carrying out the calculations:

$$\rho \sim \frac{A}{18} \text{ g cm}^{-3}.$$

The table compares the estimate against reality. Most everyday elements have atomic masses between 15 and 150, so the density estimate explains why most densities lie between 1 and 10 g cm^{-3} . It also shows why, for materials physics, cgs units are more convenient than SI units are. A typical cgs density of a solid is 3 g cm^{-3} , and 3 is a modest number and easy to remember and work with. However, a typical SI density of a solid is 3000 kg m^{-3} . Numbers such as 3000 are unwieldy. Each time you use it, you have to think, 'How many powers of ten were there again?' So the table tabulates densities using the cgs units of g cm^{-3} . I even threw a joker into the pack – water is not an element! – but the density estimate is amazingly accurate.



<i>Element</i>	$\rho_{\text{estimated}}$	ρ_{actual}
Li	0.39	0.54
H ₂ O	1.0	1.0
Si	1.56	2.4
Fe	3.11	7.9
Hg	11.2	13.5
Au	10.9	19.3
U	13.3	18.7

7.4.3 Physical interpretation

The previous method, dimensional analysis, is mostly mathematical. As a second computation of a_0 , we show you a method that is mostly physics. Besides checking the Bohr radius, it provides a physical interpretation of it. The Bohr radius is the radius of the orbit with the lowest energy (the ground state). The energy is a sum of kinetic and potential energy. This division suggests, again, a divide-and-conquer approach: first the kinetic energy, then the potential energy.