

Handout 3: Solving an Expanding Variety Model (and Notes on Quality Competition)

Some of you suggested that it was difficult to understand the role of the different components of the expanding variety model. This handout, which is heavily based on Marios' lecture notes, Daron's book and the Barro/Sala-i-Martin book, attempts to address this. The handout concentrates on the production side only, since the consumption side is the same as in the Ramsey model (there is a subtle difference in terms of the assets they hold, because there is no physical capital in this model; I will return to this later).

Structure of Production in the Expanding Variety Model

Structure of Model

The household problem is the same as in the Ramsey model (there is a subtle difference in terms of the assets they hold, because there is no physical capital in this model; I will return to this later). There are 3 sectors: (i) Final goods; (ii) Intermediate goods; (iii) R&D.

Sector	Market structure	Input	Problem and Optimization technique
Final goods	Competitive	Intermediates	$\max_{L_t, \{X_{t,j}\}_{j=0}^{N_t}} \left\{ AL_t^{1-\alpha} \left(\left[\int_0^{N_t} X_{t,j}^\varepsilon dj \right]^{\frac{1}{\varepsilon}} \right)^\alpha - w_t L_t - \int_0^{N_t} p_{t,j} X_{t,j} dj \right\}$ <p style="text-align: center;">with $\varepsilon = \alpha$</p> Technique: Lagrangian
Intermediates	Monopolistic	Final goods	$\max_{p_{t,j}} \{ p_{t,j} X_{t,j}(p_{t,j}) - \kappa(X_{t,j}) \}$ <p style="text-align: center;">with $X_{t,j}(p_{t,j}) = L_t \left(\frac{\alpha A}{p_{t,j}} \right)^{\frac{1}{1-\alpha}}$ and $\kappa(X_{t,j}) = X_{t,j}$</p> Technique: Lagrangian
R&D	Free entry	Final goods	$V_{t,j} = \Pi_{t,j} + \frac{V_{t+1,j}}{1+R_{t+1}}$ Technique: Solve stationary problem at steady state

Now we examine each sector in turn, as in the lectures.

Sector 1: Final goods

Use this sector to solve for:

- Demand function for intermediate goods
- Final output level and output growth rate in terms of growth of varieties

Optimization problem:

$$\max_{L_t, \{X_{t,j}\}_{j=0}^{N_t}} \left\{ AL_t^{1-\alpha} \int_0^{N_t} X_{t,j}^\alpha dj - w_t L_t - \int_0^{N_t} p_{t,j} X_{t,j} dj \right\}$$

FOCs:

$$w_t = (1 - \alpha) \frac{Y_t}{L_t}$$

Demand for intermediate good j :

$$p_{t,j} = \alpha A \left(\frac{L_t}{X_{t,j}} \right)^{1-\alpha} \Leftrightarrow X_{t,j} = L_t \left(\frac{\alpha A}{p_{t,j}} \right)^{\frac{1}{1-\alpha}}$$

Profits are zero from constant returns to scale.

If prices, and hence demands, for all intermediates are the same across j and over time, then

$$\begin{aligned} Y_t &= AL_t^{1-\alpha} X^\alpha \int_0^{N_t} dj \\ &= AL_t^{1-\alpha} X^\alpha N_t \\ &= AL^{1-\alpha} X^\alpha N_t \end{aligned}$$

since there is no population growth. (If there is, will there exist a balanced growth path? Output will be the product of two terms growing at potentially different rates. This is related to the "scale effect".) So output growth equals the rate of growth of varieties.

Sector 2: Intermediate goods

Use this sector to solve for:

- Prices, including monopoly markup, of intermediate goods
- Quantity demanded of intermediate goods
- Per period value of owning a blueprint

Optimization problem:

$$\max_{p_{t,j}} \left\{ p_{t,j} L_t \left(\frac{\alpha A}{p_{t,j}} \right)^{\frac{1}{1-\alpha}} - L_t \left(\frac{\alpha A}{p_{t,j}} \right)^{\frac{1}{1-\alpha}} \right\}$$

FOC:

$$\begin{aligned} L_t (\alpha A)^{\frac{1}{1-\alpha}} \cdot \left(-\frac{\alpha}{1-\alpha} p_{t,j}^{-\frac{1}{1-\alpha}-1} \right) - L_t (\alpha A)^{\frac{1}{1-\alpha}} \cdot \left(-\frac{1}{1-\alpha} p_{t,j}^{-\frac{1}{1-\alpha}-1} \right) &= 0 \\ \Leftrightarrow p_{t,j} &= \frac{1}{\alpha} \end{aligned}$$

which is constant across j and over time. $\alpha < 1$, so the monopoly markup over the input price is $\frac{1}{\alpha} - 1 > 0$ (the input is the final good and the price of this is 1). This is the monopoly distortion.

Therefore the quantity demanded of an intermediate good is

$$X_{t,j} = L_t (\alpha^2 A)^{\frac{1}{1-\alpha}}$$

and since there is no population growth, the quantity demanded is constant across j and over time:

$$X_{t,j} = xL, \text{ where } x \equiv \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}}$$

Per period value of owning a blueprint is the profit per period:

$$\begin{aligned} \Pi_{t,j} &= \frac{1}{\alpha} L_t (\alpha^2 A)^{\frac{1}{1-\alpha}} - L_t (\alpha^2 A)^{\frac{1}{1-\alpha}} \\ &= \pi L, \text{ where } \pi \equiv \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} \end{aligned}$$

Sector 3: R&D sector

Use this sector to solve for:

- Value of R&D firm with a blueprint
- Interest rate

R&D firms produce inventions ("blueprints") and then auction them off to the monopolistic intermediate goods sector. Therefore the R&D firms charge the intermediate firm a price equal to the value of owning the blueprint. This price is the value of an R&D firm which possesses a blueprint. Therefore the value of the firm is

$$V_{t,j} = \sum_{\tau=t}^{\infty} \frac{q_{\tau}}{q_t} \Pi_{t,j}$$

It solves the recursive equation:

$$\begin{aligned} V_{t,j} &= \Pi_{t,j} + \frac{V_{t+1,j}}{1 + R_{t+1}} \\ &= \pi L + \frac{V_{t+1,j}}{1 + R_{t+1}} \end{aligned}$$

Impose steady state (here, balanced growth path) so the interest rate is stationary: $R_t = R \forall t$. Then the value function is stationary:

$$V = \frac{1+R}{R} \pi L \approx \frac{\pi L}{R}$$

using the approximation $1 + R \approx 1$.

Interpretation: Consider ownership of the firm to be an asset. Ownership incurs a period opportunity cost of RV and yields a period dividend (here, profit) of πL . These must be equal in equilibrium by **no arbitrage**.

Now to determine the interest rate using the R&D production technology and **free entry**. η unit of final goods can be made into 1 blueprint (or perhaps 1 unit of the final good is made into 1 blueprint with probability $\frac{1}{\eta}$ of success and $1 - \frac{1}{\eta}$ of failure; in the latter case the final goods are wasted). The expected value to an entrepreneur of setting up an R&D firm and trying to invent a blueprint is

$$\frac{1}{\eta} V - 1$$

By free entry, this must equal zero in equilibrium, which implies:

$$V = \eta$$

Note that free entry means that the expected value of entering the R&D industry is equal to 0, NOT that the value of an R&D firm with a blueprint is 0.

Now we can solve for the interest rate using the value of an R&D firm with a blueprint together with the free entry condition:

$$\eta = \frac{\pi L}{R}$$

Rearrange:

$$\begin{aligned} R &= \frac{\pi L}{\eta} \\ &= \frac{1 - \alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} \frac{L}{\eta} \end{aligned}$$

Substitute this interest rate into the Euler condition to get the rate of growth of consumption, which in this model is also the rate of growth of output (and of varieties, as mentioned above).

Market Clearing

Impose market clearing \Rightarrow We can derive the resource constraint:

$$C_t + N_t X + \eta \Delta N_t = Y_t$$

Demand for final goods comes from its uses for consumption, as an input into the intermediate goods sector, and as an input into the R&D sector.

Note on Household Assets

Now we can just substitute this interest rate into the standard Euler equation for consumption derived in lectures. But remember, this is an economy without capital. When we set up the households, we notice that it is the same as in the Ramsey model. But if we impose market clearing in the market for assets, what must aggregate asset holdings be equal to?

All firms in the economy are capitalized on the stock market. This was true in the Ramsey economy, but since all firms made zero profits in that model, the net present value of every firm was zero and the distribution of stock holdings was irrelevant. In this economy, final goods firms still make zero profits, so the value of owning a final goods firm is still zero. Intermediate firms make zero profits over their lifetime because all of the surplus from owning a blueprint is extracted by R&D firms through the auction process. But once they own a blueprint, they have the value V . There are N_t intermediate firms. Finally, entrants into the R&D sector break even using the innovation technology and since they sell their blueprints immediately (they cannot produce intermediate goods themselves), they have zero expected value of future innovation.

Therefore the value of household assets in this economy equals the market value of blueprint-owning intermediate goods firms: $A_t = V N_t = \eta N_t$.

Distortions in this Economy

Although the intermediate goods firms make profits of zero over their lifetime, their pricing decision in each period creates a distortion because it creates a monopoly markup. This restricts the quantity demanded of intermediate goods. This in turn reduces the value of producing blueprints, and depresses the rate of growth of varieties (and therefore, from our result in the final goods sector, the growth rate of the economy).

Extensions to Other Models

This approach can be used to solve a variety of models. In fact, we can use the same approach to solve many models of quality competition (as opposed to expanding varieties), although the externalities will be different.

Notes on Quality Competition

Some of you spotted that in the model I presented in recitation, limit pricing means zero profits for (Bertrand-competing) firms in the intermediate goods sector. You were right. This means that the expression for profits in the old version of Daron's book is wrong. Limit pricing means that all surpluses from the use of higher quality intermediates are passed on by intermediate goods firms.

In this section I discuss normative implications of the model in Section 14.1 of the new version of Daron's book (which is linked from the webpage, as always). You can solve the model using exactly the approach above, because in this version the quality innovations are so large (indeed, "drastic") that owners of the blueprints have enough market power to set prices in a monopolistic manner. It is true that in quality competition models the existing intermediate goods firms (as opposed to only new firms) can carry out R&D, but since Arrow's replacement effect holds (explained at the top of page 613 of the book), only new entrants carry out R&D in equilibrium. The model can be analyzed as above. Therefore I do not write out the whole model here.

One point of difference is that in continuous time, the value of an R&D firm that possesses a blueprint follows a Hamilton-Jacobi-Bellman (HJB) equation

$$r(t)V(\nu, t|q) - \dot{V}(\nu, t|q) = \pi(\nu, t|q) - z(\nu, t|q)V(\nu, t|q),$$

which we have not covered. In discrete time, however, there is a simple Bellman equation counterpart to this equation:

$$V_{t,j} = \Pi_{t,j} + (1 - z_{j,t}) \frac{V_{t+1,j}}{1 + R_{t+1}}$$

where $z_{j,t}$ is the rate of new innovation in intermediate good j at time t (and hence the probability of a particular blueprint losing all its value).

In the exam you will not be expected to have solved this model (with HJB equations) in full before, since we did not cover it in recitation. However, you will be expected to be able to solve any similar models derived in class, and to apply the techniques obtained from solving expanding varieties models (in Marios' earlier lectures) to other models, including the quality competition model. Also, you will be expected to understand the normative implications of such models, since Marios explained this in class. You should definitely read section 14.1.5 of the book. To these implications the handout now turns.

Model in Section 14.1 of Daron's Book: Normative Implications

There are several sources of inefficiency in this economy:

- Monopoly pricing in the intermediate goods sector
- Innovators receive profits that would have gone to the displaced owners of lower quality blueprints; therefore the private return may exceed the social return to innovation
- Innovators don't receive profits after a subsequent innovator is successful in developing a higher quality blueprint, even though the innovator's original invention is what allowed the subsequent invention to take place; therefore the private return may be lower than the social return to innovation

This means that the competitive equilibrium growth rate of the economy may be higher than or lower than the Pareto optimal growth rate. An appropriate subsidy can correct the static monopoly distortion in the intermediate goods sector. How can we correct the two other externalities?

One method is to require innovators to compensate their immediate predecessor for the loss of rental income from the innovation. Then innovators do not receive profits that would have accrued to existing owners of blueprints. They do receive profits for their innovation even after their blueprint is no longer the highest quality blueprint in the economy, so they do not regard their innovation as transient. This combination solves the dynamic problems associated with "business stealing".