14.451: Introduction to Economic Growth
Problem Set 4

Due date: 10.30am, March 16, 2007.

Question 1: Consider an economy with aggregate production function

$$Y_t = AK_t^{1-\alpha}L_t^{\alpha}.$$

All markets are competitive, the labor supply is normalized to 1, capital fully depreciates after use, and the government imposes a linear tax on capital income at the rate τ, and uses the proceeds for government consumption. Consider two specifications of preferences:

1. All agents are infinitely lived, with preferences

$$\sum_{t=0}^{\infty} \beta^t \ln c_t$$

2. An overlapping generations model where agents work in the first period, and consume the capital income from their savings in the second period. The preferences of a generation born at time t, defined over consumption when young and old, are given by

$$\ln c_t^y + \beta \ln c_t^0$$

Characterize the equilibria in these two economies, and show that in the first economy, taxation reduces the level of steady state output, while in the second, it does not. Interpret this result, and in the light of this result discuss the applicability of models which try to explain income differences across countries with differences in the rates of capital income taxation.

Question 2: Consider an overlapping generations model as covered in recitation. Individuals born in any period t live for two periods. When they are young (first period, i.e. t), they work (supplying one unit of labor inelastically at the wage rate $w(t)$), consume $c_1(t)$ and save.$s(t)$. When they are old (second period, i.e. $t+1$), they just consume $c_2(t+1)$. The maximization problem for an individual born in period t is therefore

$$\max_{c_1(t),c_2(t+1),s(t)} \{\log(c_1(t)) + \beta \log(c_2(t+1))\}$$

$$\begin{aligned} \text{s.t.} \qquad c_1(t) + s(t) &\leq w(t) \\ c_2(t+1) &\leq R(t+1)s(t) \end{aligned}$$

There is no population growth so population is normalized to 1. The production function is Cobb-Douglas: $f(k(t)) = k(t)^\alpha$. The capital stock in period $t+1$ is the saving of individuals in period t. Capital fully depreciates in use.

1. Define a competitive equilibrium in this economy. Solve the optimization problem for an individual born in period t and express $c_1(t)$ and $s(t)$ in terms of $w(t)$. Use this information to derive a formula for $k(t+1)$ in terms of $k(t)$ only.

Now consider the problem of the social planner with discount factor β_S across generations. The social planner solves

$$\max_{\{c_1(t), c_2(t+1), k(t+1)\}} \sum_{t=0}^{\infty} \beta_S^t \left[\log\left(c_1(t)\right) + \beta \log\left(c_2(t+1)\right)\right]$$

$$\text{s.t.} \qquad k(t)^\alpha \geq k(t+1) + c_1(t) + c_2(t) \qquad \text{for } t = 0, 1, 2, ...$$

2. Show that the individual's Euler equation from part 1 is also a necessary condition in the solution to the social planner's problem.

3. Show that the objective function of the social planner for finite horizon T can be rewritten as

$$\log\left(c_1(0)\right) + \sum_{t=1}^{T} \beta_S^{t-1} \left[\beta_S \log\left(c_1(t)\right) + \beta \log\left(c_2(t)\right)\right] + g(T+1)$$

   for some function $g(T+1)$. In recitation we mentioned that the social planner allocation will approach a unique steady state with positive consumption. Use this information to argue that the term $g(T+1)$ goes to zero as $T \to \infty$, for any initial value of $k(0)$.

4. Prove that for a given sequence of total consumption $c(t) = c_1(t) + c_2(t)$, the social planner would choose $c_1(t) = \lambda c(t)$ and $c_2(t) = (1-\lambda)c(t)$ for constant $\lambda \in (0,1)$ (derive it). Use this information to rewrite the problem as

$$\max_{\{c_1(t), c_2(t+1), k(t+1)\}} \left\{ \log\left(c_1(0)\right) + B \sum_{t=1}^{\infty} \beta_S^{t-1} \log\left(c(t)\right) \right\}$$

$$\text{s.t.} \qquad k(t)^\alpha \geq k(t+1) + c(t) \qquad \text{for } t = 0, 1, 2, ...$$

   for constant B (derive it). Write down the Bellman equation for this problem for $t > 0$.

5. Assume that the value function in the Bellman equation takes the form $V(k) = a + b \log k$ and using this, derive the policy function $k(t+1)$ as a function of $k(t)$.

6. Derive the value of β_S (in terms of α, β, etc) for the capital evolution functions ($k(t+1)$ in terms of $k(t)$) to coincide for the competitive equilibrium and social planner allocations, for $t > 0$. Use your answers from parts 1 and 5.

Question 3: In this problem we conduct an exercise in tax smoothing/debt management. Consider an economy with infinitely-lived identical households and an infinitely-lived government. Households have the utility function

$$U_0 = \sum_{t=0}^{\infty} \beta^t \left[c_t - \frac{n_t^{1+\varepsilon}}{1+\varepsilon} \right]$$

which is linear and increasing in consumption c_t, but faces increasing and convex costs of supplying labor n_t for $\varepsilon > 0$. The sequential budget constraint of the household in period t is

$$q_t b_{t+1} + c_t = (1 - \tau_t)\, w_t n_t + b_t$$

where b_t is the asset position of the household at time t, and w_t and τ_t are the wage rate and labor tax rate at time t. Households can smooth consumption by purchasing discount bonds for the next period at price q_t. The face value of the bond at maturity is 1 unit of consumption.

Firms produce final output in each period using labor input. The production function is

$$y_t = A n_t$$

The government has a sequence of (exogenous) government expenditures $\{g_t\}_{t=0}^{\infty}$ in units of final output that must be financed in each period, either through labor taxation or by issuing government bonds (which must be redeemed).

There is no capital in this economy.

1. Write down the resource constraint for the economy in period t.

2. Write down the budget constraint for the government in period t.

3. Define the competitive equilibrium using sequential budget constraints. Add to the usual definition the satisfaction of the government budget constraint.

4. Derive labor supply as a function of the tax rate.

5. Derive the time 0 versions of the household and government budget constraints by substituting forward for b_t and using the relevant no Ponzi conditions. Use the no arbitrage condition and $b_0 = 0$ to simplify your expressions.

6. Now we consider the Ramsey problem for this economy. The social planner chooses allocations to maximize utility U_0 subject to the constraint that the allocation is a competitive equilibrium. Set up the problem at time 0 using the versions of the constraints derived in part 5. Prove that the household budget constraint is redundant given the resource and government budget constraints being satisfied with equality (which will be the case here).

7. Use your answer in part 4 to write the Ramsey problem in terms of τ_t and g_t only. Prove that it is optimal for the government to set the labor tax rate to a constant value, i.e.

$$\tau_t = \tau^* \quad \forall\, t = 0, 1, 2, ...$$

Give the equation that implicitly determines τ^* as a function of the sequence $\{g_t\}_{t=0}^{\infty}$. Note: I do not want Lagrange multipliers in this equation.