14.451 Lecture Notes 1

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1 A portfolio problem

To set the stage, consider a simple finite horizon problem.

A risk averse agent can invest in two assets:

- riskless asset (bond) pays gross return 1

- risky asset (stock) pays gross return R: random variable with support $\left[\underline{R}, \overline{R}\right]$, $\underline{R} > 0$

Preferences: expected utility

$$\mathrm{E}\left[u\left(c\right)\right].$$

Agent has wealth w needs to choose how much to invest in stocks, s, and how much in bonds, b.

$$b + s = w.$$

Suppose agent cannot "short" the stock, $s \geq 0$, and cannot borrow, $b \geq 0$.

One period problem. Random consumption stream

$$c = Rs + b.$$

The problem to solve is:

$$\max_{s \in [0,w]} \mathrm{E}\left[u\left(Rs + w - s\right)\right].$$

Standard concave problem. If interior solution then

$$\mathrm{E}\left[\left(R - 1\right) u'\left(Rs + w - s\right)\right] = 0.$$

Do it for two periods, $t = 0, 1, 2$. (At $t = 0, 1$ agent makes investment decisions, at $t = 2$ agent just consumes). In periods $t = 1, 2$ the random return on stocks R_t is drawn independently from the same distribution.

Choose how much to invest in the first period s_0, b_0 and how much to invest in the second period, conditional on what happened in the first period, $s_1\left(R_1\right), b_1\left(R_1\right)$. The problem to solve is:

$$\max_{\substack{s_0 \geq 0, b_0 \geq 0 \\ s_1(R_1) \geq 0, b_1(R_1) \geq 0}} \mathrm{E}\left[u\left(c\left(R_1, R_2\right)\right)\right]$$

subject to the constraints

$$
\begin{aligned}
b_0 + s_0 &= w_0 \\
w_1 (R_1) &= R_1 s_0 + b_0 \\
b_1 (R_1) + s_1 (R_1) &= w_1 (R_1) \\
c (R_1, R_2) &= R_2 s_1 (R_1) + b_1 (R_1)
\end{aligned}
$$

Do it for T periods, it gets complicated as the investment decisions are conditioned on all past realizations of R.

Idea: define the best you can do from period t on, then go backward.

Important step: choose the right *state variable*. A state variable is the right summary for all that happened before t.

Here: wealth w_t.

Let's go back to two period example. Suppose you enter period 1 with any possible wealth w. The best you can do gives you expected utility

$$
V_1 (w) = \max_{s \in [0, w]} E \left[u (Rs + w - s) \right].
$$

Now go back to $t = 0$ and solve

$$
\max_{s \in [0, w_0]} E \left[V_1 (Rs + w_0 - s) \right].
$$

Assume preferences are:

$$
u (c) \equiv \frac{1}{1 - \gamma} c^{1 - \gamma}
$$

$$
V_1 (w) = \max_{s \in [0, w]} E \left[\frac{1}{1 - \gamma} (Rs + w - s)^{1 - \gamma} \right].
$$

We can now make the change of variable

$$
\theta = s / w,
$$

which is the fraction of wealth invested in stocks. Then

$$
\begin{aligned}
V_1 (w) &= \max_{\theta \in [0, 1]} E \left[\frac{1}{1 - \gamma} (R\theta + 1 - \theta)^{1 - \gamma} w^{1 - \gamma} \right] \\
&= \left\{ \max_{\theta \in [0, 1]} E \left[\frac{1}{1 - \gamma} (R\theta + 1 - \theta)^{1 - \gamma} \right] \right\} w^{1 - \gamma} \\
&= \frac{\xi}{1 - \gamma} w^{1 - \gamma}
\end{aligned}
$$

where

$$
\xi \equiv (1 - \gamma) \max_{\theta \in [0, 1]} E \left[\frac{1}{1 - \gamma} (R\theta + 1 - \theta)^{1 - \gamma} \right]
$$

(it's better to be careful and not simplify the two $(1-\gamma)$ in this expression, because $(1-\gamma)$ may be negative and then the maximization problem would go upside down!)

Optimal allocation is to invest a constant fraction θ^* in stocks, in both periods.

But this is true also for any (finite) horizon T. The value function for the T-period problem is:

$$V_t(w) = \frac{\xi^{T-t-1}}{1-\gamma} w^{1-\gamma}.$$

The optimal policy is to invest a fraction θ^* of wealth in stocks each period.

2   An optimal saving problem

We now turn to a non-stochastic problem. The main differences with the previous example are that consumption happens in all periods, there is discounting, and the risk free rate is not zero.

Objective:

$$\sum_{t=0}^{T} \beta^t u(c_t)$$

Flow constraint:

$$a_t = (1+r) a_{t-1} + y - c_t.$$

Terminal condition:

$$a_T \geq 0.$$

State variable: a_{t-1}

$$V_t(a_{t-1}) = \max \sum_{j=t}^{T} \beta^j u(c_{t+j}) \ \ s.t....$$

Recursive approach: start from

$$V_T(a) = u((1+r) a + y)$$

and go back from here.

Going to infinite horizon objective is

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

now the problem is independent of how far we are from T: stationarity. We can hope to define a stationary value function

$$V(a_t) = \max \sum_{j=t}^{\infty} \beta^j u(c_{t+j}) \ \ s.t....$$

and attack it with the right recursive methods.

This is the class of problems we will study now, with five essential ingredients: infinite horizon, discrete time, deterministic setup, discounting, stationarity.

3   A general recursive problem

We will work now on a general deterministic problem.
Ingredients:

- Infinite horizon

- Discrete time

- Discounting at rate $\beta \in (0,1)$

- State variable x_t in some set X

- Constraint correspondence $\Gamma : X \to X$

$$x_{t+1} \in \Gamma\left(x_t\right)$$

- Payoff function
$$F\left(x_t, x_{t+1}\right)$$

  defined for all $x_t \in X$ and all $x_{t+1} \in \Gamma\left(x_t\right)$. In other words, with $A \equiv \{(x,y) \in X \times X : y \in \Gamma\left(x\right)\}$, we have a function $F : A \to R$.

A *plan* is a sequence $\{x_t\}_{t=0}^{\infty}$.
A *feasible plan from* x_0, is a plan with $x_{t+1} \in \Gamma\left(x_t\right)$. Set of feasible plans denoted $\Pi\left(x_0\right)$.

Objective is to find a sequence $\{x_t\}_{t=0}^{\infty}$ that maximizes the discounted sum of payoffs
$$\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

This needs to be defined for all feasible plans.
An *optimal plan from* x_0 is a plan that achieves the maximum.
Assumption 1. $\Gamma\left(x\right)$ non-empty $\forall x \in X$
Assumption 2.
$$\lim_{T \to \infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) \text{ exists}$$

for all feasible plans from any initial $x_0 \in X$.

4   Principle of Optimality

On the one hand we have the sequence problem (SP):

$$V^*(x_0) \equiv \max \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

subject to

$$x_{t+1} \in \Gamma(x_t) \qquad t = 0, 1, ...$$

and x_0 given. Or more compactly, $V^*(x_0) = \max_{x \in \Pi(x_0)} u(x)$. Solving this problem is finding optimal plans x^* that attain the value V^* for all initial conditions.

On the other hand we have the Bellman equation

$$V(x) = \max_{x \in \Gamma(x)} \{F(x, y) + \beta V(y)\},$$

which is a functional equation (FE). Solving this problem is finding a V that satisfies this equation.

Define also the policy correspondence

$$G(x) \equiv \arg \max_{x \in \Gamma(x)} \{F(x, y) + \beta V^*(y)\}$$

(note: this is the maximum on the right hand side of the Bellman equation but using the specific function $V = V^*$).

The principle of optimality is about the relationship between these two problems, about the relationship between V solving FE and V^* defined by SP. It is also about the relationship between optimal plans x^* for SP and plans generated using the policy correspondence G.

Theorem 4.2. Suppose $V^*(x)$ is well defined for all $x \in X$ then V^* satisfies the Bellman equation:

$$V^*(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V^*(y)\}.$$

Proof. Take any $x_0 \in X$. Let $\{x_t^*\}_{t=0}^{\infty}$ be an optimal plan from x_0. By definition

$$V^*(x_0) = F(x_0, x_1^*) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t^*, x_{t+1}^*) \geq F(x_0, x_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1})$$

for all plans $\{x_t\}_{t=0}^{\infty}$ which are feasible from x_0. Taking any $x_1 \in \Gamma(x_0)$ let $\{x_t\}_{t=1}^{\infty}$ be an optimal plan from x_1 so

$$V^*(x_1) = \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1}).$$

Then we have

$$F(x_0, x_1^*) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t^*, x_{t+1}^*) \geq F(x_0, x_1) + \beta V^*(x_1)$$

for all $x_1 \in \Gamma(x_0)$. Using this at $x_1 = x_1^*$ we have

$$\sum_{t=1}^{\infty} \beta^{t-1} F(x_t^*, x_{t+1}^*) \geq V^*(x_1^*),$$

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but since $\{x_t^*\}_{t=1}^\infty$ is a feasible plan starting at x_1^* this must hold as an equality. So we have

$$F(x_0, x_1^*) + \beta V^*(x_1^*) \geq F(x_0, x_1) + \beta V^*(x_1) \text{ for all } x_1 \in \Gamma(x_0).$$

∎

Note: In SLP they proceed without assuming that a maximum exists; with SP defined by a supremum. This is preferable. The argument is very similar.

We have established that $V = V^*$ solves the Bellman equation. Next we provide a result showing that optimal plans must be generated by the policy correspondence.

Theorem 4.4. Suppose $V^*(x)$ is well defined for all $x \in X$. Suppose the plan $\{x_t^*\}_{t=0}^\infty$ is optimal from x_0, then

$$V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*) \qquad t = 0, 1, ...$$

Proof. Use same calculations as for Theorem 4.2 and apply induction. We showed that

$$V^*(x_0) = F(x_0, x_1^*) + \beta V^*(x_1^*)$$

and we showed that $\{x_t^*\}_{t=1}^\infty$ is an optimal plan from x_1^*. Since $\{x_t^*\}_{t=1}^\infty$ is optimal from x_1^*, we can proceed along the same reasoning to show that

$$V^*(x_1^*) = F(x_1^*, x_2^*) + \beta V^*(x_2^*)$$

and that $\{x_t^*\}_{t=2}^\infty$ is optimal from x_2^*. Continuing in this way establishes the result for all $t = 0, 1, ...$ ∎

This theorem implies that an optimal plan satisfies $x_{t+1}^* \in G(x_t^*)$. Is the reverse always true? Is a plan generated by the policy function necessarily optimal? No, there are counterexamples as the next example illustrates.

Example (from SLP). Take

$$F(x, y) = x - \beta y$$

$$X = \{x \in \mathbb{R} \text{ and } x \geq 0\}$$

$$\Gamma(x) = [0, \beta^{-1}x].$$

Economically, this corresponds to a savings problem for a consumer with initial wealth x_0 that has linear utility over consumption $c_t = x_t - \beta x_{t+1}$ and faces a market gross interest rate $R = \beta^{-1}$. Intuitively, the consumer is indifferent to many plans for consumption. In particular, consuming immediately ($x_t = 0$ for $t = 1, 2, ...$) is optimal and $V^*(x_0) = x_0$. Consuming everything in the second period is also optimal ($x_1 = \beta^{-1}x_0$ and $x_t = 0$ for $t = 2, 3, ...$) which involves maximal savings in the very first period. Many other plans are optimal. This multiplicity of solutions is also reflected in policy correspondence which is

$$G(x) = \arg\max_{y \in \Gamma(x)} \{x - \beta y + \beta y\} = \Gamma(x).$$

However, the path $x_t = \beta^{-t}x_0$ that is generated from G by setting $x_{t+1} = \beta^{-1}x_t$ for all $t = 0, 1, ...$is clearly not optimal since then $c_t = F(x_t, x_{t+1}) = 0$ for all $t = 0, 1, ...$and $u(x) = 0 < V^*(x_0) = x_0$. The consumer is willing to postpone for any number of periods, but not forever.

We need an extra condition to rule out plans that essentially never deliver. The condition turns out to be a limiting condition that is much like a "No-Ponzi" condition for values $V^*(x_t)$ along the proposed path.

We now strengthen the previous result with such a condition and provide the converse.

Theorems 4.4 and 4.5 (Ivan Werning's version). A feasible plan $\{x_t^*\}_{t=0}^{\infty}$ is optimal if and only if

$$V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*) \text{ for } t = 0, 1, ...$$

and

$$\lim_{t \to \infty} \beta^t V^*(x_t^*) = 0.$$

Proof. (Necessity) We already showed the first part. We need only show the limit condition. Substituting $V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*)$ repeatedly we arrive at

$$V^*(x_0) = \sum_{t=0}^{T} F(x_t^*, x_{t+1}^*) + \beta^{T+1} V^*(x_{T+1}^*).$$

Notice that the limit $\lim_{T \to \infty} \sum_{t=0}^{T} F(x_t^*, x_{t+1}^*)$ exists and is equal to $V^*(x_0)$. Therefore, the limit $\lim_{t \to \infty} \beta^t V^*(x_t^*)$ must exist and be equal to zero.

(Sufficiency) The converse is true by exactly the same calculations: since we have that $\lim_{t \to \infty} \beta^t V^*(x_t^*) = 0$ it follows that $\sum_{t=0}^{T} F(x_t^*, x_{t+1}^*)$ has a well defined limit, equal to $V^*(x_0)$. This also show that $\{x_t^*\}_{t=0}^{\infty}$ is optimal. ∎

Now we want to prove that a solution to FE is V^*. The problem is that this is generally not always the case. There may be other solutions as the next example shows.

Example. Take

$$F(x, y) = x - \beta y$$

$$X = \mathbb{R}$$

and set $\Gamma(x) = \{-\beta^{-1}x\}$ if $x > 0$ and $\Gamma(x) = \{\beta^{-1}x\}$ otherwise. Thus, there is only one feasible plan for any x_0 and $V^*(x_0) = \max\{2x, 0\}$. Note that $\beta^t V^*(x_t) \to 0$ for the only feasible path. But what about the Bellman equation? There is another solution with $V(x) = x$. Note that for this solution it is not true that $\lim \beta^t V(x_t) = -x_0$ for all feasible paths.

The previous example suggests that other solutions do not satisfy a limiting condition that V^* does satisfy. We thus strengthen the requirements.

Theorem 4.3 (Ivan's version). Suppose V is a finite valued function that solves FE. Suppose for each $x_0 \in X$ there exists a feasible plan $\{x_t^*\}_{t=0}^{\infty}$ with

$V(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*)$ for $t = 0, 1, 2, ...$ and $\lim \beta^t V(x_t^*) = 0$. Suppose also that

$$\limsup_{t \to \infty} \beta^t V(x_t) \geq 0$$

for all feasible plans with $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) > -\infty$. Then $V = V^*$ defined from the SP.

Proof. We want to show that for any $x_0 \in X$

$$V(x_0) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

for all plans that are feasible from x_0. Take any $x_0 \in X$ and find a plan $\{x_t^*\}_{t=0}^{\infty}$ that satisfies the theorem's hypothesis. The inequality above is immediately true if $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = -\infty$. Otherwise, iterating on the Bellman equation gives us

$$V(x_0) = \sum_{t=0}^{T-1} \beta^t F(x_t^*, x_{t+1}^*) + \beta^T V(x_T^*) \geq \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T V(x_T).$$

Taking the supremum limit on both sides gives

$$V(x_0) = \sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) + \limsup_{t \to \infty} \beta^T V(x_T)$$

which gives the desired inequality because $\limsup_{t \to \infty} \beta^T V(x_T) \geq 0$. This completes the proof since x_0 was arbitrary. ∎

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