14.384 Time Series Analysis, Fall 2007
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Recitation 3

Filtering

In lecture 4, we introduced filtering. Here we'll spend a bit more time deriving some common filters and showing how to use them. Recall that an ideal band-pass filter has

$$B^*(e^{i\omega}) = \begin{cases} 1 & \lambda \in [\pi_l, \pi_h] \\ 0 & \text{otherwise} \end{cases}$$

and can be written as

$$B^*(e^{i\omega}) = \sum_{-\infty}^{\infty} \beta_j^* e^{-i\omega j}$$

where

$$\begin{aligned} \beta_j^* = & \frac{1}{2\pi} \int_{|\omega| \in [\pi_l, \pi_h]} e^{i\omega j} d\omega \\ = & \frac{1}{2\pi} \int_{\pi_l, \pi_h} e^{i\omega j} d\omega + \int_{-\pi_h, -\pi_l} e^{i\omega j} d\omega \\ = & \frac{1}{2\pi i j} \left(e^{i\pi_h j} - e^{i\pi_l j} + e^{-i\pi_l j} - e^{-i\pi_h j} \right) \\ = & \begin{cases} \frac{\sin(j\pi_h) - \sin(j\pi_l)}{\pi j} & j \neq 0 \\ \frac{\pi_h - \pi_l}{\pi} & j = 0 \end{cases} \end{aligned}$$

Baxter-King

Baxter and King (1999) proposed approximating the ideal filter with one of order J by solving

$$\begin{aligned} & \min_{B()} \frac{1}{2\pi} \int_{-\pi}^{\pi} |B(e^{i\omega} - B^*(e^{i\omega})|^2 d\omega \\ & \text{s.t.} \\ & B(1) = \phi \end{aligned}$$

where the constraint may or may not be present. We might want to impose $B(1) = 0$ so that the filtered series is stationary, or if we're constructing a low-pass filter, we might want $B(1) = B(e^{i0}) = 1$ to preserve the lowest frequency movements.

The Lagrangian is

$$L = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(B^*(e^{i\omega} - \sum_{|j|\leq J} b_j e^{-i\omega j}\right)\left(B^*(e^{i\omega} - \sum_{|j|\leq J} b_j e^{-i\omega j}\right)' d\omega + \lambda(\sum_{|j|\leq J} b_j - \phi)$$

The first order conditions are

$$[b_k] : 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(B^*(e^{i\omega} - \sum_{|j|\leq J} b_j e^{-i\omega j}\right) e^{i\omega k} d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\omega k}\left(B^*(e^{i\omega} - \sum_{|j|\leq J} b_j e^{-i\omega j}\right)' d\omega + \lambda$$

$$[\lambda] : 0 = \sum_{|j|\leq J} \beta_j - \phi$$

Using the fact that $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(j-k)} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} B^*(\omega)e^{i\omega j} = \beta_j^*$, the first order conditions for b_j are

$$b_j = \beta_j^* + \frac{\lambda}{2}$$

Using the constraint to solve for λ gives:

$$\phi = \sum_{|j|\leq J} \beta_j^* + \frac{2J+1}{2}\lambda$$

$$\lambda = \frac{2}{2J+1}\left(\phi - \sum_{|j|\leq J} \beta_j^*\right)$$

To summarize: the Baxter King filter of order J on $[\pi_l, \pi_h]$ constrained to have $B(0) = \phi$ is given by

$$b_j = \beta_j^* + \theta$$

where

$$\beta_j^* = \begin{cases} \frac{\sin(j\pi_h) - \sin(j\pi_l)}{\pi j} & j \neq 0 \\ \frac{\pi_h - \pi_l}{\pi} & j = 0 \end{cases}$$

$$\theta = \frac{1}{2J+1}\left(\phi - \sum_{|j|\leq J} \beta_j^*\right)$$

Christiano-Fitzgerald

Christiano and Fitzgerald (1999) propose a generalization of the Baxter-King filter. The advocate choosing a finite approximation to the ideal filter by solving

$$\min_{B_t()} E[(B_t(L)x_t - B^*(L)x_t)^2]$$

where x_t is some chosen process. $B_t(L)$ is allowed to use all data available in your sample of length T. Note that $B_t(L)$ will generally not be symmetric and will change with t. As shown in problem set 1, this is equivalent to solving,

$$\min_{B_t()} \frac{1}{2\pi} \int_{-\pi}^{\pi} |B_t(e^{i\omega}) - B^*(e^{i\omega})|^2 S_x(\omega) d\omega$$

where $S_x(\omega)$ is the spectrum of x_t. The Baxter-King filter can be considered a special case of this approach where x_t is white noise, and we restrict $B_t()$ to be time-invariant and only have $b_j \neq 0$ for $|j| \leq J$. Christiano and Fitzgerald argue that having x_t a random walk works well for macro time-series.

Hodrick-Prescott

Recall that the Hodrick-Prescott filter solves:

$$\min_{\tau_t} \{ \sum (\tau_t - x_t)^2 + \lambda(\tau_{t+1} - 2\tau_t + \tau_{t-1})^2 \}$$

For $1 < t < T - 1$, the first order conditions are:

$$\begin{aligned} 0 = 2(\tau_t - x_t) + 2\lambda(& - 2(\tau_{t+1} - 2\tau_t + \tau_{t-1}) + \\ & + \tau_t - 2\tau_{t-1} + \tau_{t-2} + \\ & + \tau_{t+2} - 2\tau_{t+1} + \tau_t) \end{aligned}$$

$$0 = -x_t + \lambda\tau_{t-2} - 4\lambda\tau_{t-1} + (6\lambda + 1)\tau_t - 4\lambda\tau_{t+1} + \lambda\tau_{t+2}$$

The first order conditions for $t = 0, 1, T - 1, T$ are similar. Writing them all in matrix form, we have

$$\tau = \begin{bmatrix} \lambda & -2\lambda + 1 & \lambda & 0 & 0 & \ldots & 0 \\ -2\lambda & 5\lambda + 1 & -4\lambda & \lambda & 0 & \ldots & 0 \\ \lambda & -4\lambda & 6\lambda + 1 & -4\lambda & \lambda & \ldots & 0 \\ \vdots & & & \ddots & & & \end{bmatrix}^{-1} X$$

We can then form our cycle as $c_t = x_t - \tau_t$.