14.30 EXAM 3 - SUGGESTED ANSWERS

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Question 1

(i) *False/Uncertain.* MLE produces estimators that are consistent, but may or may not be unbiased.

(ii) *False/Uncertain.* MM estimators may be biased or unbiased, consistent or inconsistent.

(iii) *False.* The first moment of the sample mean is $E\left(\overline{X}\right)$, which will equal the population mean in general, whether or not the sample is normally distributed: $E\left(\overline{X}\right) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E\left(X_i\right) = E\left(X_i\right)$.

(iv) *False/Uncertain.* The power function is defined as

$$\begin{aligned} \pi\left(\theta \mid \delta\right) &= \alpha_\theta\left(\delta\right) \text{ if } \theta \in \Omega_0 \\ &= 1 - \beta_\theta\left(\delta\right) \text{ if } \theta \in \Omega_1 \end{aligned}$$

Thus for any given θ, either $\pi\left(\theta \mid \delta\right) = \alpha_\theta\left(\delta\right)$ or $1 - \pi\left(\theta \mid \delta\right) = \beta_\theta\left(\delta\right)$, but not both, so the definition of the power function does not tell us anything about $\alpha_\theta\left(\delta\right) + \beta_\theta\left(\delta\right)$.

(v) *False.* The interval is the random variable, not the parameter. So when you estimate a confidence interval at some probability level α, it means that with probability α, your estimated interval contains the true parameter value.

(vi) *False.* The sample variance is a random variable that we can calculate, $S^2 = \frac{1}{n-a}\sum\left(X_i - \overline{X}\right)^2$. The variance of the sample mean is a parameter, $V\left(\overline{X}\right) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}V\left(X_i\right) = \frac{\sigma^2}{n}$.

Question 2

(i) In general, $L\left(a \mid x_1, x_2\right) = f\left(x_1\right) f\left(x_2\right)$, and $f\left(x_i\right) = \frac{1}{a}$ if $x_i = 0$, $f\left(x_i\right) = 1 - \frac{1}{a}$ if $x_i = 1$. From this, we can find $L\left(a \mid x_1, x_2\right)$ for different combinations of $\left(x_1, x_2\right)$.

$L\left(a \mid 0, 0\right) = f\left(0\right) f\left(0\right) = \frac{1}{a^2}$
$L\left(a \mid 1, 0\right) = f\left(1\right) f\left(0\right) = \left(1 - \frac{1}{a}\right)\frac{1}{a}$
$L\left(a \mid 0, 1\right) = f\left(0\right) f\left(1\right) = \frac{1}{a}\left(1 - \frac{1}{a}\right)$

$L(a \mid 1,1) = f(1)\, f(1) = \left(1 - \frac{1}{a}\right)^2$

(ii) If $(x_1, x_2) = (0,1)$, we want to pick a to maximize $\frac{1}{a}\left(1 - \frac{1}{a}\right)$, or equivalently, $\ln\left(\frac{a-1}{a^2}\right) = \ln(a-1) - 2\ln a$. We take the derivative of the log-likelihood function with respect to a and set it equal to zero to get the first order condition, and then solve for a:

$$\begin{aligned} \frac{1}{a-1} &= \frac{2}{a} \\ a &= 2a - 2 \\ \widehat{a}_{MLE} &= 2 \end{aligned}$$

Then we can use the invariance property to get $\widehat{a^2}_{MLE} = (\widehat{a}_{MLE})^2 = 4$.

(iii) If $(x_1, x_2) = (1,1)$, we want to maximize $\left(\frac{a-1}{a}\right)^2$ or $2\ln(a-1) - 2\ln(a)$. When we take the derivative we find

$$\frac{2}{a-1} - \frac{2}{a} > 0 \text{ for all } a$$

which implies that our MLE estimator is not finite. In other words, for any real number a that we might substitute into the likelihood function, the first derivative will be greater than zero, so the likelihood function is increasing, and a is not the maiximizing value. Putting this together with the invariance property suggests that $\widehat{\left(\frac{1}{a}\right)}_{MLE} = 0$.

(iv) The first population moment is $E(X) = 0\left(\frac{1}{a}\right) + 1\left(1 - \frac{1}{a}\right) = \frac{a-1}{a}$. To compute the MM estimator, we will set this equal to the sample mean and solve for a. For our sample, $(x_1, x_2) = (0,1)$, so $\overline{X} = \frac{1}{2}$.

$$\begin{aligned} \frac{a-1}{a} &= \frac{1}{2} \\ 2a - 2 &= a \\ \widehat{a}_{MM} &= 2 \end{aligned}$$

Question 3

(i) We have an independent, normally distributed sample of 25 with a sample mean of 4 and sample variance of 9. We want to create a symmetric 95% confidence interval for the population mean. Thus we want to find random variables $A\left(\overline{X}, S^2\right)$ and $B\left(\overline{X}, S^2\right)$ such that

$$\Pr\left(A\left(\overline{X}, S^2\right) \leq \mu \leq B\left(\overline{X}, S^2\right)\right) = 0.95$$

or equivalently,

$$\Pr\left(A\left(\overline{X}, S^2\right) > \mu\right) = 0.025 \text{ and } \Pr\left(B\left(\overline{X}, S^2\right) < \mu\right) = 0.025$$

We know that $\sqrt{n}\frac{\overline{X}-\mu}{S} \sim t_{n-1}$, which implies that

$$\begin{aligned}
\Pr\left(\sqrt{n}\frac{\overline{X}-\mu}{S} > t_{0.025,n-1}\right) &= 0.025\text{and } \Pr\left(\sqrt{n}\frac{\overline{X}-\mu}{S} < -t_{0.025,n-1}\right) = 0.025 \\
\Pr\left(\overline{X} - \frac{S}{\sqrt{n}}t_{0.025,n-1} > \mu\right) &= 0.025 \text{ and } \Pr\left(\overline{X} + \frac{S}{\sqrt{n}}t_{0.025,n-1} < \mu\right) = 0.025
\end{aligned}$$

So our 95% confidence interval is $\overline{X} \pm \frac{S}{\sqrt{n}}t_{0.025,n-1}$, and in the problem at hand, we have $4 \pm \frac{3}{5}2.064$, and the confidence interval is $(2.76, 5.24)$.

(ii) Now we want to test

$$\begin{aligned}
H_0 &: \mu \leq 10 \\
H_1 &: \mu > 10
\end{aligned}$$

We want a decision rule such that we reject H_0 if $\overline{X} > k$, and $\Pr(\text{Reject } H_0 \mid \mu = 9.7) = 0.05$. So we will choose k to satisfy

$$\Pr\left(\overline{X} > k \mid \mu = 9.7\right) = 0.05$$

We again use the fact that $\sqrt{n}\frac{\overline{X}-\mu}{S} \sim t_{n-1}$, so that we have

$$\begin{aligned}
\Pr\left(\sqrt{n}\frac{\overline{X}-9.7}{S} > t_{0.05,n-1} \mid \mu = 9.7\right) &= 0.05 \\
\Pr\left(\overline{X} > 9.7 + \frac{S}{\sqrt{n}}t_{0.05,n-1} \mid \mu = 9.7\right) &= 0.05
\end{aligned}$$

and $k = 9.7 + \frac{S}{\sqrt{n}}t_{0.05,n-1} = 9.7 + \frac{3}{5}(1.711) = 10.7266$, so we reject H_0 if $\overline{X} > 10.7266$. We have $\overline{X} = 4$, thus we fail to reject the null.

(iii) The size of the test is the maximum probability of type I error for any value of $\mu \leq 10$:

$$\alpha = \sup_{\mu \leq 10} \alpha_\mu$$

Because $\overline{X} \sim \mu + \frac{S}{\sqrt{n}}(t_{n-1})$, increasing μ will shift up the distribution of $\overline{X}$ and $\Pr\left(\overline{X} > 10.7266 \mid \mu\right)$ will be increasing in μ for a fixed sample. Thus our maximum type I error will occur for the largest $\mu \in \Omega_0$, or when $\mu = 10$. Then

$$\begin{aligned}
\Pr\left(\overline{X} > 10.7266 \mid \mu = 10\right) &= \Pr\left(5\frac{\overline{X}-10}{3} > 5\frac{10.7266-10}{3} \mid \mu = 10\right) \\
&= \Pr(t_{24} > 1.211)
\end{aligned}$$

We know from our table that $\Pr(t_{24} > 1.381) = 0.10$, so the size of the test will be greater than 10%.

(iv) Now we want to test

$$\begin{aligned} H_0 &: \mu = 0 \\ H_1 &: \mu \neq 0 \end{aligned}$$

at the 1% level. So we want to find k_1 and k_2 such that we reject H_0 if $\overline{X} \notin [k_1, k_2]$, and $\Pr\left(\overline{X} < k_1 \mid \mu = 0\right) = 0.005 = \Pr\left(\overline{X} > k_2 \mid \mu = 0\right)$. Again we use $\sqrt{n}\frac{\overline{X}-\mu}{S} \sim t_{n-1}$, so that we have

$$\begin{aligned} \Pr\left(\sqrt{n}\frac{\overline{X}-0}{S} > t_{0.005,n-1} \mid \mu = 0\right) &= 0.005 \\ \Pr\left(\overline{X} > \frac{S}{\sqrt{n}} t_{0.005,n-1} \mid \mu = 0\right) &= 0.005 \end{aligned}$$

and similarly,

$$\begin{aligned} \Pr\left(\sqrt{n}\frac{\overline{X}-0}{S} < -t_{0.005,n-1} \mid \mu = 0\right) &= 0.005 \\ \Pr\left(\overline{X} < -\frac{S}{\sqrt{n}} t_{0.005,n-1} \mid \mu = 0\right) &= 0.005 \end{aligned}$$

So in this case we have $k_1 = -\frac{S}{\sqrt{n}} t_{0.005,n-1} = -\frac{3}{5} 2.797 = -1.6782$, and $k_2 = 1.6782$. Since $\overline{X} = 4$ is out of this range, we reject the null hypothesis that the diet has no effect on weight. Since we are easily able to reject at the 1% level, we must have a p-value that is less than 1%.

(v) Now because we are assuming that the population variance is known, we can use the standard normal distribution (and the Z table) instead of the t distribution. We want to set up a test of the equality of population means (let the mean population weight loss for Dr. Diet be μ_1, for Dr. BetterDiet be μ_2):

$$\begin{aligned} H_0 &: \mu_1 = \mu_2 \\ H_1 &: \mu_1 \neq \mu_2 \end{aligned}$$

or equivalently,

$$\begin{aligned} H_0 &: \mu_1 - \mu_2 = 0 \\ H_1 &: \mu_1 - \mu_2 \neq 0 \end{aligned}$$

Because both populations are normally distributed (and we assume independence), we know that $\overline{X}_1 - \overline{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$. We want a critical region such that we reject if $\overline{X}_1 - \overline{X}_2 \notin [k_1, k_2]$, and $\Pr(\text{Reject } H_0 \mid \mu_1 - \mu_2 = 0) =$

0.05. We know that

$$\Pr\left(\frac{\overline{X}_1-\overline{X}_2-0}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}>Z_{0.025}\mid\mu_1-\mu_2=0\right) = 0.025$$

$$\Pr\left(\overline{X}_1-\overline{X}_2>Z_{0.025}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\mid\mu_1-\mu_2=0\right) = 0.025$$

and similarly

$$\Pr\left(\frac{\overline{X}_1-\overline{X}_2-0}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}<-Z_{0.025}\mid\mu_1-\mu_2=0\right) = 0.025$$

$$\Pr\left(\overline{X}_1-\overline{X}_2<-Z_{0.025}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\mid\mu_1-\mu_2=0\right) = 0.025$$

So we have $k_1=-Z_{0.025}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}=-\sqrt{\frac{9}{25}+\frac{9}{50}}1.96=-\sqrt{\frac{27}{50}}1.96=-1.44$, and $k_2=1.44$. So we should reject if $\overline{X}_1-\overline{X}_2\notin[-1.44,1.44]$. Since $\overline{X}_1-\overline{X}_2=-1$, we fail to reject.

(vi) We could reject the null hypothesis of identical effects if $\overline{X}_1-\overline{X}_2<-1.44$, or equivalently, we could reject if $\frac{\overline{X}_1-\overline{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}<-1.96$. In either case, the average weight loss for Dr. BetterDiet would have to be more than 1.44 pounds greater than the average weight loss for Dr. Diet. In other words, if we had $\overline{X}_2>5.44$ and $\overline{X}_1=4$, we would reject the null.