Homework #4 Solutions

Problem 1

a. The lower bound is 1. If n is even, let X be $((c,c),...,(b,b)...)$, where (c,c) is played for $n/2$ periods and (b,b) is played for $n/2$ periods. For n odd, $X=((c,c),...,(b,b),...,(a,a))$, where (c,c) is played for $(n-1)/2$ periods, (b,b) is played for $(n-1)/2$ periods. The nash equilibrium strategy is to play X as long as X has been played in every previous period, and otherwise play (c,c) for the rest of the game. This is a nash equilibrium because there is no possible deviation for either player. If a player deviates, he will get payoff at most 1 in every future period, so the best he can get by deviating is payoff n, which he is indifferent to.

b. For n even, $X_n=((c,c),...,(b,b),...,(a,a))$, where (c,c) is played for $n/2$ periods, (b,b) is played for $n/2-1$ periods. If n is odd, let X be $((c,c),...,(b,b)...)$, where (c,c) is played for $(n-1)/2$ periods and (b,b) is played for $(n+1)/2$ periods. For $n=1$, the strategy is to play (b,b), for payoff 2. We prove by induction.

Suppose that for $n<T$, there is a subgame perfect equilibrium with payoff $n+1$. At $n=T$, the subgame perfect equilibrium is to play X_T as long as everyone has played on the equilibrium path. If a player deviates from playing (c,c) at some period with t rounds remaining, we play X_t as punishment. If a player deviates from playing (b,b) or (a,a), we continue on X_T.

A player that deviates with t rounds remaining gets payoff 1 in that round, and then plays the X_t subgame perfect equilibrium with payoff $t+1$, for a total of $t+2$. This will never exceed $T+1$, so on histories on the equilbrium path there are no profitable deviations. There are no deviations after histories when we are on X_t because they are subgame perfect equilbria, by the inductive hypothesis.

Problem 2

a. This is never SPE. Player 2 has payoff 0 in equilibrium, so he can always deivate to R for payoff 1.

b. This is also never SPE, for the same reason.

c. This is always a SPE. In every period, players are playing a stage game nash equilibrium, so the strategy is subgame perfect equilibrium.

Problem 3

(a) Suppose for each cycle, (C,C) is played a times and (D,D) is played b times. Then average payoff for the cycle is $\frac{5a+b}{a+b}$. To make $1.1<\frac{5a+b}{a+b}<1.2$, we need $19a<b<39a$. Let's choose $a=1,\ b=20$. The strategy profile is for each player, play D for $t=21k+i$, for $i=1,2,\ldots,20$ and play C for $t=21k$ if no deviation has occurred. If any deviation has occured, play D forever.

No player has incentive to deviate when some player has deviated since (D,D) is NE of the stage game. When a player is supposed to play D at $t=21k+i$, to prevent deviation we need

$$6+1\cdot\frac{\delta}{1-\delta}\leq 1\cdot\frac{1}{1-\delta}+4\delta^{21-i}\frac{1}{1-\delta^{21}}$$

note that the right side of inequality is minimized at $i=1$, so we only need to check that case. For $\delta=0.999$, it holds.

When a player is supposed to play C, to prevent deviation we need

$$6+1\cdot\frac{\delta}{1-\delta}\leq 1\cdot\frac{1}{1-\delta}+4\cdot\frac{1}{1-\delta^{21}}$$

For $\delta=0.999$, it holds.

(b) The strategy profile is that player 1 plays D for $t=4k+i$, for $i=0,1,2$ and plays C for $t=4k+3$ and player 2 plays C for all t if no deviation has occurred. If any deviation has occured, play D forever. When (D,C) is supposed to be played, player 1 has no incentive to deviate as he gets the maximum possible payoff. For player 2, we only need to check $t=4k$ case (similar logic from part a) as if she were to deviate she would have maximum incentive at that case. To prevent deviation we need

$$1\cdot\frac{1}{1-\delta}\leq\delta^{3}\cdot 5\cdot\frac{1}{1-\delta^{3}}$$

for $\delta=0.999$, it holds. When (C,C) is supposed to be played, for player 1 we need $6+1\cdot\frac{\delta}{1-\delta}\leq 6\cdot\frac{1}{1-\delta}-\frac{1}{1-\delta^{3}}$ and for player 2 we need $6+1\cdot\frac{\delta}{1-\delta}\leq 5\cdot\frac{1}{1-\delta^{3}}$. Both holds for $\delta=0.999$.

(c) The answer is no. To give player 1 the average payoff of more than 5.8, we have to give player to the average payoff of less than 1. Since player 2 can get at least 1 by deviation and can get at least 1 in all static NE, we cannot construct SPE where player 2 gets less than 1 on average.

No player has incentive to deviate when some player has deviated since (D,D) is NE of the stage game.

Problem 4

(a) If $|p_1-p_2|<c$, we have an interior solution: there is a "mid-point" x^* such that $0<x^*<1$ and kid at x^* is indifferent. In other words,

$$cx^*+p_1=c\left(1-x^*\right)+p_2$$

so $x^*=\frac{1}{2}+\frac{p_2-p_1}{2c}$. If $|p_1-p_2|\geq c$, then all kids go to one firm.

To start, we find one stage (static) NE. If $|p_1-p_2|\geq c$, it cannot be an equilibrium as higher price firm makes zero and has incentive to cut its price so that it can make positive profit. For $|p_1-p_2|<c$, firm 1 solves

$$max_{p_1}p_1\left\{\frac{1}{2}+\frac{p_2-p_1}{2c}\right\}$$

Taking FOC, you get $p_1^{BR}(p_2)=\frac{c+p_2}{2}$. Similirly, $p_2^{BR}(p_1)=\frac{c+p_1}{2}$. Thus, $p_1=p_2=c$ as NE.

Since this NE is a unique SPE for the stage game, playing this NE for all period is a unique SPE for finite games.

(b-1) Check what would be the best response if the other firm plays p^*. If $p^*-c\geq\frac{p^*+c}{2}$ (or $p^*\geq 3c$), then BR is to charge p^*-c and capture the whole market. In this case, to make sure that there is no incentive to deviate, we need

$$\frac{p^*}{2}\left(1+\delta+\delta^2+\cdots\right)\quad\geq\quad\left(p^*-c\right)+\frac{c}{2}\left(\delta+\delta^2+\cdots\right)$$

$$
\begin{aligned}
\frac{p^*}{2} &\geq (p^*-c)(1-\delta)+\frac{c\delta}{2} \\
(2\delta-1)\,p^* &\geq (3\delta-2)\,c
\end{aligned}
$$

Note that $\hat{p}=c$ here.

If $\delta>\frac{1}{2}$, then $p^*\geq\frac{3\delta-2}{2\delta-1}c$. Thus, maximum p^* is $\bar{p}$.

If $\delta=\frac{1}{2}$, then $0\geq-2c$ works for all p^*, so maximum p^* is $\bar{p}$.‘

If $\delta<\frac{1}{2}$, then $p^*\leq\frac{2-3\delta}{1-2\delta}c=p_{max}$.

If $\delta\geq\frac{1}{3}$, $p_{max}\geq 3c$ so maximum p^* is $\bar{p}$.

If $\delta<\frac{1}{3}$, $p_{max}<3c$ so the best responce is $\frac{p^*+c}{2}$. In this case, to make sure that there is no incentive to deviate, we need

$$
\begin{aligned}
\frac{p^*}{2}\quad 1+\delta+\delta^2+\cdots &\geq \frac{(p^*+c)^2}{8c}+\frac{c}{2}\quad \delta+\delta^2+\cdots \\
\frac{p^*}{2} &\geq \frac{(p^*+c)^2}{8c}(1-\delta)+\frac{c\delta}{2}
\end{aligned}
$$

Solving, we get

$$p^*\leq\frac{1+3\delta}{1-\delta}c$$

(b-2) Suppose the firm makes u_0 by deviating during the war period. Note that u_0 is zero if $\hat{p}\leq 0$ and positive if $\hat{p}>0$.

Let V_0 as a sum of current and future profit at the war period. Then $V_0=\frac{\hat{p}}{2}+\frac{p^*}{2}\frac{\delta}{1-\delta}$ and to prevent deviation during the war state, we need

$$\frac{1}{1-\delta}V_0\geq u_0+\frac{\delta}{1-\delta}V_0$$

or $V_0\geq u_0$. Since $u_0\geq 0$, the best punishment is to choose $V_0=u_0=0$. This could be done by choosing $\hat{p}=-\frac{\delta}{1-\delta}p^*$.

For collusion period, to prevent deviation we need

$$
\begin{aligned}
\frac{p^*}{2} &\geq (1-\delta)(p^*-c)+\delta\cdot 0 \\
\left(\frac{1}{2}-\delta\right)p^* &\leq (1-\delta)\,c
\end{aligned}
$$

Thus, if $\delta\geq\frac{1}{2}$, then $p^*=\bar{p}$. If $\delta<\frac{1}{2}$, then $p^*=min\left\{\bar{p},\ \frac{(1-\delta)c}{\frac{1}{2}-\delta}\right\}$.