14.06 Lecture Notes
Intermediate Macroeconomics

George-Marios Angeletos
MIT Department of Economics

Spring 2004

Chapter 6

Endogenous Growth I: AK, H, and G

6.1 The Simple AK Model

6.1.1 Pareto Allocations

- Total output in the economy is given by

$$Y_t = F(K_t, L_t) = AK_t,$$

where $A > 0$ is an exogenous parameter. In intensive form,

$$y_t = f(k_t) = Ak_t.$$

- The social planner's problem is the same as in the Ramsey model, except for the fact that output is linear in capital:

$$\max \sum_{t=0}^{\infty} u(c_t)$$

$$s.t. \quad c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

- The Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(1 + A - \delta)$$

Assuming CEIS, this reduces to

$$\frac{c_{t+1}}{c_t} = [\beta\,(1 + A - \delta)]^{\theta}$$

or

$$\ln c_{t+1} - \ln c_t \approx \theta(R - \rho)$$

where $R = A - \delta$ is the net social return on capital. That is, consumption growth is proportional to the difference between the real return on capital and the discount rate. Note that now the real return is a constant, rather than diminishing with capital accumulation.

- Note that the resource constraint can be rewritten as

$$c_t + k_{t+1} = (1 + A - \delta)k_t.$$

Since total resources (the RHS) are linear in k, an educated guess is that optimal consumption and investment are also linear in k. We thus propose

$$c_t = (1 - s)(1 + A - \delta)k_t$$
$$k_{t+1} = s(1 + A - \delta)k_t$$

where the coefficient s is to be determined and must satisfy $s \in (0, 1)$ for the solution to exist.

- It follows that

$$\frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{y_{t+1}}{y_t}$$

so that consumption, capital and income all grow at the same rate. To ensure perpetual growth, we thus need to impose

$$\beta(1 + A - \delta) > 1,$$

or equivalently $A - \delta > \rho$. If that condition were not satisfied, and instead $A - \delta < \rho$, then the economy would shrink at a constant rate towards zero.

- From the resource constraint we then have

$$\frac{c_t}{k_t} + \frac{k_{t+1}}{k_t} = (1 + A - \delta),$$

implying that the consumption-capital ratio is given by

$$\frac{c_t}{k_t} = (1 + A - \delta) - [\beta(1 + A - \delta)]^\theta$$

Using $c_t = (1 - s)(1 + A - \delta)k_t$ and solving for s we conclude that the optimal saving rate is

$$s = \beta^\theta (1 + A - \delta)^{\theta - 1}.$$

Equivalently, $s = \beta^\theta(1 + R)^{\theta-1}$, where $R = A - \delta$ is the net social return on capital. Note that the saving rate is increasing (decreasing) in the real return if and only if the EIS is higher (lower) than unit, and $s = \beta$ for $\theta = 1$. Finally, to ensure $s \in (0, 1)$, we impose

$$\beta^\theta (1 + A - \delta)^{\theta - 1} < 1.$$

This is automatically ensured when $\theta \leq 1$ and $\beta(1 + A - \delta) > 1$, as then $s = \beta^\theta (1 + A - \delta)^{\theta-1} \leq \beta < 1$. But when $\theta > 1$, this puts an upper bound on A. If A exceeded that upper bound, then the social planner could attain infinite utility, and the problem is not well-defined.

- We conclude that

Proposition 24 *Consider the social planner's problem with linear technology* $f(k) = Ak$ *and CEIS preferences. Suppose* $(\beta, \theta, A, \delta)$ *satisfy* $\beta(1+A-\delta) > 1 > \beta^\theta (1+A-\delta)^{\theta-1}$. *Then, the economy exhibits a balanced growth path. Capital, output, and consumption all grow at a constant rate given by*

$$\frac{k_{t+1}}{k_t} = \frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = [\beta(1+A-\delta)]^\theta > 1.$$

while the investment rate out of total resources is given by

$$s = \beta^\theta (1+A-\delta)^{\theta-1}.$$

The growth rate is increasing in productivity A*, increasing in the elasticity of intertemporal substitution* θ*, and decreasing in the discount rate* ρ *(where* $\beta = \frac{1}{1+\rho}$*).*

- Differences in productivities and preferences may thus help explain differences, not only in the level of output and the rate of investment, but also in growth rates.

6.1.2 The Frictionless Competitive Economy

- Consider now how the social planner's allocation is decentralized in a competitive market economy.

- Suppose that the same technology that is available to the social planner is available to each single firm in the economy. Then, the equilibrium rental rate of capital and the equilibrium wage rate will be given simply

$$r = A \quad \text{and} \quad w = 0.$$

- The arbitrage condition between bonds and capital will imply that the interest rate is

$$R = r - \delta = A - \delta.$$

- Finally, the Euler condition for the household will give

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1+R\right)\right]^{\theta}.$$

- We conclude that the competitive market allocations coincide with the Pareto optimal plan. Note that this is true only because the private and the social return to capital coincide.

6.1.3 What is next

- The analysis here has assumed a single type of capital and a single sector of production. We next consider multiple types of capital and multiple sectors. In essence, we "endogenize" the capital K and the productivity A – for example, in terms of physical versus human capital, intentional capital accumulation versus unintensional spillovers, innovation and knowledge creation, etc. The level of productivity and the growth rate will then depend how the economy allocates resources across different types of capital and different sectors of production. What is important to keep in mind from the simple AK model is the importance of linear returns for delivering perpetual growth.

6.2 A Simple Model of Human Capital

- We now consider a variant of the AK model, where there are two types of capital, physical (or tangible) and human (or intangible). We start by assuming that both types of capital are produced with the same technology, that is, the absorb resources in the same intensities. We later consider the case that the production of human capital is more intensive in time/effort/skills than in machines/factories.

6.2.1 Pareto Allocations

- Total output in the economy is given by

$$Y_t = F(K_t, H_t) = F(K_t, h_t L_t),$$

where F is a neoclassical production function, K_t is aggregate capital in period t, h_t is human capital per worker, and $H_t = h_t L_t$ is effective labor.

- Note that, due to CRS, we can rewrite output per capita as

$$y_t = F(k_t, h_t) = F\left(\frac{k_t}{h_t}, 1\right) \frac{h_t}{k_t + h_t}[k_t + h_t] =$$

or equivalently

$$y_t = F(k_t, h_t) = A\left(\kappa_t\right)[k_t + h_t],$$

where $\kappa_t = k_t/h_t = K_t/H_t$ is the ratio of physical to human capital, $k_t + h_t$ measures total capital, and

$$A\left(\kappa\right) \equiv \frac{F\left(\kappa, 1\right)}{1+\kappa} \equiv \frac{f(\kappa)}{1+\kappa}$$

represents the return to total capital.

- Total output can be used for consumption or investment in either type of capital, so that the resource constraint of the economy is given by

$$c_t + i_t^k + i_t^h \leq y_t.$$

The laws of motion for two types of capital are

$$\begin{aligned} k_{t+1} &= (1-\delta_k)k_t + i_t^k \\ h_{t+1} &= (1-\delta_h)h_t + i_t^h \end{aligned}$$

As long as neither i_t^k nor i_t^h are constrained to be positive, the resource constraint and the two laws of motion are equivalent to a single constraint, namely

$$c_t + k_{t+1} + h_{t+1} \le F(k_t, h_t) + (1-\delta_k)k_t + (1-\delta_h)h_t$$

- The social planner's problem thus becomes

$$\max \sum_{t=0}^{\infty} u(c_t)$$
$$s.t. \quad c_t + k_{t+1} + h_{t+1} \le F(k_t, h_t) + (1-\delta_k)k_t + (1-\delta_h)h_t$$

- Since there are two types of capital, we have two Euler conditions, one for each type of capital. The one for physical capital is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left[1 + F_k(k_{t+1}, h_{t+1}) - \delta_k\right],$$

while the one for human capital is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left[1 + F_h(k_{t+1}, h_{t+1}) - \delta_h\right].$$

- Combining the two Euler condition, we infer

$$F_k(k_{t+1}, h_{t+1}) - \delta_k = F_h(k_{t+1}, h_{t+1}) - \delta_h.$$

Remember that F is CRS, implying that both F_k and F_h are functions of the ratio $\kappa_{t+1} = k_{t+1}/h_{t+1}$. In particular, F_k is decreasing in κ and F_h is increasing in κ. The above condition therefore determines a unique optimal ratio κ^* such that

$$\frac{k_{t+1}}{h_{t+1}} = \kappa_{t+1} = \kappa^*$$

for all $t \geq 0$. For example, if $F(k,h) = k^\alpha h^{1-\alpha}$ and $\delta_k = \delta_h$, then $\frac{F_k}{F_h} = \frac{\alpha}{1-\alpha}\frac{h}{k}$ and therefore $\kappa^* = \frac{\alpha}{1-\alpha}$. More generally, the optimal physical-to-human capital ratio is increasing in the relative productivity of physical capital and decreasing in the relative depreciation rate of physical capital.

- Multiplying the Euler condition for k with $k_{t+1}/(k_{t+1}+h_{t+1})$ and the one for h with $h_{t+1}/(k_{t+1}+h_{t+1})$, and summing the two together, we infer the following "weighted" Euler condition:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left\{1 + \frac{k_{t+1}[F_k(k_{t+1},h_{t+1}) - \delta_k] + h_{t+1}[F_h(k_{t+1},h_{t+1}) - \delta_h]}{k_{t+1}+h_{t+1}}\right\}$$

By CRS, we have

$$F_k(k_{t+1},h_{t+1})k_{t+1} + F_h(k_{t+1},h_{t+1})h_{t+1} = F(k_{t+1},h_{t+1}) = A\left(\kappa_{t+1}\right)\left[k_{t+1}+h_{t+1}\right]$$

It follows that

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left\{1 + A\left(\kappa_{t+1}\right) - \frac{\delta_k k_{t+1} + \delta_h h_{t+1}}{k_{t+1}+h_{t+1}}\right\}$$

Using the fact that $\kappa_{t+1} = \kappa^*$, and letting

$$A^* \equiv A\left(\kappa^*\right) \equiv \frac{F\left(\kappa^*,1\right)}{1+\kappa^*}$$

represent the "effective" return to total capital and

$$\delta^* \equiv \delta\left(\kappa^*\right) \equiv \frac{\kappa^*}{1+\kappa^*}\delta_k + \frac{1}{1+\kappa^*}\delta_h$$

the "effective" depreciation rate of total capital, we conclude that the "weighted" Euler condition evaluated at the optimal physical-to-human capital ratio is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left[1 + A^* - \delta^*\right].$$

- Assuming constant elasticity of intertemporal substitution, namely $u(c) = \frac{c^{1-1/\theta}}{1-1/\theta}$, this reduces to

$$\frac{c_{t+1}}{c_t} = [\beta (1 + A^* - \delta^*)]^\theta$$

or

$$\ln c_{t+1} - \ln c_t \approx \theta(A^* - \delta^* - \rho)$$

where $A^* - \delta^*$ is the net social return to total savings. Note that the return is constant along the balanced growth path, but it is not exogenous. It instead depends on the ratio of physical to human capital. The latter is determined optimally so as to maximize the net return on total savings. To see this, note that $k_{t+1}/h_{t+1} = \kappa^*$ indeed solves the following problem

$$\max \;\; F(k_{t+1}, h_{t+1}) - \delta_k k_{t+1} - \delta_h h_{t+1}$$
$$s.t. \quad k_{t+1} + h_{t+1} = constant$$

Equivalently, κ^* maximizes the return to savings:

$$\kappa^* = \arg\max_\kappa \left[1 + A(\kappa) - \delta(\kappa)\right]$$

- Given the optimal ratio κ^*, the resource constraint can be rewritten as

$$c_t + [k_{t+1} + h_{t+1}] = (1 + A^* - \delta^*)[k_t + h_t].$$

Like in the simple Ak model, an educated guess is then that optimal consumption and total investment are also linear in total capital:

$$c_t = (1 - s)(1 + A^* - \delta^*)[k_t + h_t],$$
$$k_{t+1} + h_{t+1} = s(1 + A^* - \delta^*)[k_t + h_t].$$

The optimal saving rate s is given by

$$s = \beta^{\theta} \left(1 + A^{*} - \delta^{*}\right)^{\theta - 1}.$$

- We conclude that

Proposition 25 *Consider the social planner's problem with CRS technology $F(k, h)$ over physical and human capital and CEIS preferences. Let*

$$A(\kappa) \equiv \frac{F(\kappa, 1)}{1+\kappa} \quad \textit{and} \quad \delta(\kappa) \equiv \frac{\kappa}{1+\kappa}\delta_k + \frac{1}{1+\kappa}\delta_h.$$

Next, let

$$\kappa^{*} = \arg\max_{\kappa} \left[1 + A(\kappa) - \delta(\kappa)\right]$$

and suppose $(\beta, \theta, F, \delta_k, \delta_h)$ satisfy $\beta\left[1 + A(\kappa^{*}) - \delta(\kappa^{*})\right] > 1 > \beta^{\theta}\left[1 + A(\kappa^{*}) - \delta(\kappa^{*})\right]^{\theta-1}$. Then, the economy exhibits a balanced growth path. Physical capital, human capital, output, and consumption all grow at a constant rate given by

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \left\{\beta\left[1 + A(\kappa^{*}) - \delta(\kappa^{*})\right]\right\}^{\theta} > 1.$$

while the investment rate out of total resources is given by $s = \beta^{\theta}\left[1 + A(\kappa^{*}) - \delta(\kappa^{*})\right]^{\theta-1}$ and the optimal ratio of physical to human capital is $k_{t+1}/h_{t+1} = \kappa^{*}$. The growth rate is increasing in the productivity of either type of capital, increasing in the elasticity of intertemporal substitution, and decreasing in the discount rate.

6.2.2 Market Allocations

- Consider now how the social planner's allocation is decentralized in a competitive market economy.

- The household budget is given by

$$c_t + i_t^k + i_t^h + b_{t+1} \leq y_t + (1 + R_t)b_t.$$

and the laws of motion for the two types of capital are

$$\begin{aligned} k_{t+1} &= (1 - \delta_k)k_t + i_t^k \\ h_{t+1} &= (1 - \delta_h)h_t + i_t^h \end{aligned}$$

We can thus write the household budget as

$$c_t + k_{t+1} + h_{t+1} + b_{t+1} \leq (1 + r_t - \delta_k)k_t + (1 + w_t - \delta_h)h_t + (1 + R_t)b_t.$$

Note that $r_t - \delta_k$ and $w_t - \delta_h$ represent the market returns to physical and human capital, respectively.

- Suppose that the same technology that is available to the social planner is available to each single firm in the economy. Then, the equilibrium rental rate of capital and the equilibrium wage rate will be given simply

$$r_t = F_k(\kappa_t, 1) \quad \text{and} \quad w_t = F_h(\kappa_t, 1),$$

where $\kappa_t = k_t/h_t$.

- The arbitrage condition between bonds and the two types of capital imply that

$$R_t = r_t - \delta_k = w_t - \delta_h.$$

Combining the above with the firms' FOC, we infer

$$\frac{F_k(\kappa_t, 1)}{F_h(\kappa_t, 1)} = \frac{r_t}{w_t} = \frac{\delta_h}{\delta_k}$$

and therefore $\kappa_t = \kappa^*$, like in the Pareto optimum. It follows then that

$$R_t = A^* - \delta^*,$$

where $A^* = A(\kappa^*)$ and $\delta^* = \delta(\kappa^*)$, as above.

- Finally, the Euler condition for the household is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(1 + R_t)\,.$$

Using $R_t = A^* - \delta^*$, we conclude

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = [\beta\,(1 + A^* - \delta^*)]^{\theta}$$

- Hence, the competitive market allocations once again coincide with the Pareto optimal plan. Note that again this is true only because the private and the social return to *each* type of capital coincide.

6.3 Learning by Education (Ozawa and Lucas)

- The benefit of accumulating human capital is that it increases labor productivity. The cost of accumulating human capital is that it absorbs resources that could be used in the production of consumption goods or physical capital.

- The previous analysis assumed that human capital is produced with the same technology as consumption goods and physical capital. Perhaps a more realistic assumption is that the production of human capital is relative intensive in time and effort. Indeed, we can think of formal education as a choice between how much time to allocate to work (production) and how much to learning (education).

notes to be completed

6.4 Learning by Doing and Knowledge Spillovers (Arrow and Romer)

6.4.1 Market Allocations

- Output for firm m is given by

$$Y_t^m = F\left(K_t^m, h_t L_t^m\right)$$

where h_t represents the aggregate level of human capital or knowledge. h_t is endogenously determined in the economy (we will specify in a moment how), but it is taken as exogenous from either firms or households.

- Firm profits are given by

$$\Pi_t^m = F\left(K_t^m, h_t L_t^m\right) - r_t K_t^m - w_t L_t^m$$

The FOCs give

$$\begin{aligned} r_t &= F_K\left(K_t^m, h_t L_t^m\right) \\ w_t &= F_L\left(K_t^m, h_t L_t^m\right) h_t \end{aligned}$$

Using the market clearing conditions for physical capital and labor, we infer $K_t^m/L_t^m = k_t$, where k_t is the aggregate capital labor ratio in the economy. We conclude that, given k_t and h_t, market prices are given by

$$\begin{aligned} r_t &= F_K(k_t, h_t) = f'(\kappa_t) \\ w_t &= F_L(k_t, h_t)h_t = [f(\kappa_t) - f'(\kappa_t)\kappa_t]h_t \end{aligned}$$

where $f(\kappa) \equiv F(\kappa, 1)$ is the production function in intensive form and $\kappa_t = k_t/h_t$.

- Households, like firms, take w_t, r_t and h_t as exogenously given. The representative household maximizes utility subject to the budget constraint

$$c_t + k_{t+1} + b_{t+1} \leq w_t + (1 + r_t - \delta)k_t + (1 + R_t)b_t.$$

Arbitrage between bonds and capital imply $R_t = r_t - \delta$ and the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left(1 + R_t\right) = \beta(1 + r_t - \delta).$$

- To close the model, we need to specify how h_t is determined. Following Arrow and Romer, we assume that knowledge accumulation is the unintentional by-product of learning-by-doing in production. We thus let the level of knowledge to be proportional to either the level of output, or the level of capital:

$$h_t = \eta k_t,$$

for some constant $\eta > 0$.

- It follows that the ratio $k_t/h_t = \kappa_t$ is pinned down by $\kappa_t = 1/\eta$. Letting the constants A and ω be defined

$$A \equiv f'(1/\eta) \quad \text{and} \quad \omega \equiv f(1/\eta)\eta - f'(1/\eta),$$

we infer that equilibrium prices are given by

$$r_t = A \quad \text{and} \quad w_t = \omega k_t.$$

Substituting into the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left(1 + A - \delta\right).$$

Finally, it is immediate that capital and output grow at the same rate as consumption. We conclude

Proposition 26 *Let $A \equiv f'(1/\eta)$ and $\omega \equiv f(1/\eta)\eta - f'(1/\eta)$, and suppose $\beta(1+A-\delta) > 1 > \beta^{\theta}(1+A-\delta)^{\theta-1}$. Then, the market economy exhibits a balanced growth path. Physical capital, knowledge, output, wages, and consumption all grow at a constant rate given by*

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = [\beta(1+A-\delta)]^{\theta} > 1.$$

The wage rate is given by $w_t = \omega k_t$, while the investment rate out of total resources is given by $s = \beta^{\theta}(1+A-\delta)^{\theta-1}$.

6.4.2 Pareto Allocations and Policy Implications

- Consider now the Pareto optimal allocations. The social planner recognizes that knowledge in the economy is proportional to physical capital and internalizes the effect of learning-by-doing. He thus understands that output is given by

  $$y_t = F(k_t, h_t) = A^* k_t$$

  where $A^* \equiv f(1/\eta)\eta = A + \omega$ represents the *social* return on capital. It is therefore as if the social planner had access to a linear technology like in the simple Ak model, and therefore the Euler condition for the social planner is given by

  $$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1+A^*-\delta).$$

- Note that the social return to capital is higher than the private (market) return to capital:

  $$A^* > A = r_t$$

  The difference is actually ω, the fraction of the social return on savings that is "wasted" as labor income.

Proposition 27 *Let* $A^* \equiv A+\omega \equiv f(1/\eta)\eta$ *and suppose* $\beta\left(1+A^*-\delta\right)>1>\beta^\theta\left(1+A^*-\delta\right)^{\theta-1}$. *Then, the Pareto optimal plan exhibits a balanced growth path. Physical capital, knowledge, output, wages, and consumption all grow at a constant rate given by*

$$\frac{y_{t+1}}{y_t}=\frac{c_{t+1}}{c_t}=\left[\beta\left(1+A^*-\delta\right)\right]^\theta>1.$$

Note that $A<A^*$, *and therefore the market growth rate is lower than the Pareto optimal one.*

- *Exercise:* Reconsider the market allocation and suppose the government intervenes by subsidizing either private savings or firm investment. Find, in each case, what is the subsidy that implements the optimal growth rate. Is this subsidy the optimal one, in the sense that it maximizes social welfare?

6.5 Government Services (Barro)

notes to be completed